THE

PRINCIPLES

OF

MECHANICAL PHILOSOPHY

APPLIED TO

INDUSTRIAL MECHANICS:

FORMING A SEQUEL TO THE AUTHOR'S

"EXERCISES ON MECHANICS AND NATURAL PHILOSOPHY."

BY THOMAS TATE, F.R.A.S.,

Turner

OF KNELLER TRAINING COLLEGE, TWICKENHAM;
LATE MATHEMATICAL PROFESSOR AND LECTURER ON CHEMISTRY IN THE
NATIONAL SOCIETY'S TRAINING COLLEGE, BATTERSEA.

AUTHOR OF
"THE PRINCIPLES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS";
"A TREATISE ON THE STRENGTH OF MATERIALS";
"THE PRINCIPLES OF GEOMETRY AND MENSURATION, ETC."
ETC. ETC.

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TO

G. TATE, F. G. S.,

PRESIDENT OF THE BERWICKSHIRE NATURALISTS' CLUB, SECRETARY OF THE ALNWICK SCIENTIFIC INSTITUTION, ETC.

DEAR BROTHER,

THIS Work I inscribe to you as a tribute of esteem and affection, and as a memorial of those joyous days of our youth, when, hand in hand, we were engaged in the disinterested pursuit of knowledge.

That the evening of your life may be as serene and happy as its meridian has been active and useful, is the earnest wish of

Your affectionate Brother,

THOMAS TATE.

Kneller Hall, Isleworth,
February, 28. 1853.

PREFACE.

THE present work is intended to give a concise yet comprehensive exposition of the principles of mechanical philosophy, as applied to industrial mechanics. With the view of rendering the subject instructive to practical engineers and teachers, the investigations are, for the most part, conducted on algebraic and geometrical principles, and numerous applications and illustrations are given throughout the work. With respect to the importance of industrial mechanics, as a branch of national education, Professor Moseley observes, in his report on the Hydraulic machines of the Great Exhibition: —

"In reporting upon the hydraulic machines exhibited, it is impossible to refrain from adverting to the general neglect of those elementary principles of scientific knowledge on which the perfection of such machines always depends, and in some cases, their whole usefulness, in an economical point of view. The Exhibition affords positive evidence of the sacrifice of a large amount of capital, and of much mechanical ingenuity, due simply to the ignorance of certain acknowledged principles of hydraulic science. In adverting to this fact, the jury cannot but observe that the success with which the principles of mechanical science, in their application to practical questions, are beginning to be cultivated in France,

appears in the superiority of French hydraulic machines. Thus their water-wheels have obtained a perfection which is probably nowhere else to be found. The total amount of such power derivable from the running waters of France, and applicable to manufacturing purposes, has been largely increased by expedients of a scientific character. Among the most remarkable of these is the introduction, now almost universal in France, of the curved float-boards of M. Poncelet, in undershot and breast-wheels, and of the turbine of M. Fourneyron. It is not, however, only in the adoption of new forms of water-wheels in France, that the improvement has been apparent, but in the better establishment and more skilful working of the old forms, such as are in use in this country."

In the dynamical portion of this work, the Author has availed himself of the researches of Poncelet, Morin, and Moseley; and in the portion on hydraulics, of the experimental labours of Eytelwein, Bossut, and others. At the same time the Author flatters himself, that the scientific man will find considerable portions of the work original, not only in matter, but also in the methods of investigation. In particular he would call attention to the following portions:—

Art. **51.**, containing a new demonstration of the parallelogram of forces; Arts. **106.** and **109.**, containing new formulæ for finding the centres of gravity of irregular surfaces and solids; Art. **121.**, containing a simple method for graduating the safety valve; Art. **161.**, containing a new and expeditious method for determining the point of rupture of an arch; Chapter II,, on various general formulæ relative to work; Art. **228.**, &c., on certain general propositions relative to the motion of a body on an inclined plane, the friction being given; Art. **45.**, &c., page 45., containing various new formulæ relative to the flow of water in pipes, and in open

channels; Art. 71., &c., page 310., on the work of hydraulic engines, including reaction wheels and the centrifugal pump; Art. 99., &c., page 330., containing various general formulæ relative to the work of steam; Art. 116., p. 341., containing the demonstration of a new and simple law relative to the conditions of the maximum work of steam; Art. 260., &c., relative to the modulus of machines; Art. 252., &c., on the work accumulated in the parts of machines; and Art. 151., containing the generalisation of various practical problems.

T. TATE.

February 28. 1853.

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PRINCIPLES

OF

MECHANICAL PHILOSOPHY

APPLIED TO

INDUSTRIAL MECHANICS.

PART I.

FUNDAMENTAL PRINCIPLES AND PRELIMINARY INVESTIGATIONS RELATIVE TO WORK, ETC.

CHAPTER I.

GENERAL VIEWS AND DEFINITIONS RELATIVE TO MATTER, FORCE, AND MOTION. PROPERTIES OF MATTER. MEASURE OF FORCES. LAWS OF MOTION.

METHOD OF PHILOSOPHY.

1. THE regularity of the various phenomena of the material world leads us to infer, that matter is governed by certain invariable laws. Bodies everywhere fall to the earth's surface in vertical lines, and at the same place they descend through equal spaces in equal times. A pendulum, at the same place, always completes its vibrations in the same time. The various machines constructed by human intelligence invariably perform the same amount of work when acting under the same conditions. Day and night, summer and winter, the flux and reflux of the tides of the ocean, the different aspects of the planetary bodies,—all take place with unvarying uniformity and harmony. Guided by induction and calculation, philosophers have determined that throughout the vast expanse of the visible universe there is not an atom of matter which is not subject to the laws of force and motion. The

4, 2

discovery of these laws constitutes the primary object of Mechanical Philosophy.

Observation and experiment are the only means whereby we can attain a true knowledge of the facts of Nature, and then, by a process of induction, we are enabled to arrive at those general laws and principles upon which these facts depend. Having ascended by a process of induction, we then descend by a process of deduction, tracing, by the aid of mathematical analysis, the application of these general laws and principles to the explanation and elucidation of phenomena which at first lay beyond the sphere of our intelligence. The coincidence of our theoretical deductions with observed facts, must be regarded as the highest confirmation of the truth of the principles upon which such deductions are based.

DEFINITIONS.

2. MATTER is known to us by its properties, which affect our senses. The MASS of a body is the quantity of matter which it contains. The DENSITY of a body is the comparative quantity of matter contained in a given volume.

3. A body is in MOTION when it is in the act of changing its place. When a body passes over equal spaces in equal successive portions of time, its motion is said to be *uniform*. When the successive spaces, described in equal times, continually increase, the motion is said to be *accelerated;* and to be *retarded* when those spaces continually decrease. Motion is *uniformly accelerated* or *retarded* when the increments or decrements of the spaces, passed over in equal successive portions of time, are always equal. The VELOCITY of a body is measured by the space uniformly passed over in a given time: one second is usually taken as the given time, and the space described is measured in feet. When the motion of a body is accelerated or retarded, the velocity is not measured by the space actually passed over in a given time, but by the space which would have been passed over in the given time, if the motion had continued uniform from that point. The MOMENTUM of a body is its quantity of motion.

4. FORCE is that which produces, or tends to produce, motion in a body, or it is that which changes the uniform and rectilinear motion of a body. Thus, pressure, impact, gravity, &c., are called forces. When a force acts only for an instant, it is called *impulsive*, and when it acts without intermission, it is called a *constant*

force. Constant forces may be either *uniform* or *variable*. A force is uniform when it always produces equal effects in *equal* successive portions of time; and it is *variable* when the effects produced in equal portions of time are *unequal*.

5. MECHANICAL PHILOSOPHY treats of the laws of rest and motion of material bodies. It is divided into four branches. (1.) STATICS, which treats of the equilibrium of solid bodies. (2.) DYNAMICS, which treats of the motion of solid bodies. (3.) HYDROSTATICS, which treats of the equilibrium of fluid bodies. (4.) HYDRODYNAMICS, which treats of the motion of fluid bodies.

6. Forces are known to us only by the effects which they produce. In order, therefore, to estimate the magnitude of forces, we must compare the effects which they produce under the same circumstances. A force may be estimated by the PRESSURE which it produces upon some obstacle, or it may be estimated by the motion which it produces in a body in a given time. In the former case, the measure of the force is said to be STATICAL, and in the latter case DYNAMICAL.

PROPERTIES OF MATTER.

7. The properties of matter are usually divided into *primary* or *essential*, and *secondary* or *non-essential*. The former are those properties without which we cannot conceive matter to exist; the latter are those which, depending upon the particular laws impressed upon different substances, do not necessarily enter into our abstract conceptions of matter: thus, for example, had it pleased the Creator, the law of gravitation might have been different from what it is; or in the place of the law of perfect elasticity, observed in some bodies, all the forms of matter might have been practically incompressible. It is obvious, therefore, that the secondary properties of matter could not have become known to us anterior to observation and experiment. The relative adaptation of these secondary, or *contingent*, properties of matter to the conditions and constitution of the universe, affords us the most striking evidence of the existence and attributes of a great and intelligent cause.

The primary properties of matter are the following: —

8. EXTENSION, or that property whereby every body must occupy a certain limited space. We necessarily conceive every body to have length, breadth, and thickness.

9. IMPENETRABILITY, or that property whereby no two substances can occupy the same space at the same instant of time.

The most important secondary properties of matter, considered in relation to mechanical philosophy, are the following: —

10. COMPRESSIBILITY AND EXPANSIBILITY are those properties by virtue of which bodies may be made to occupy a smaller or a larger space. The susceptibility to compression shows that all bodies must contain pores, or spaces between the ultimate particles or atoms of which they are composed, and that there is no substance in nature which is absolutely solid.

In consequence of these properties, bodies differ very much in their density. Of two bodies, A and a, having equal volumes, that body is the more dense which contains the greater mass or quantity of matter: thus, if M and m represent the masses of A and a respectively, and D and d their respective densities, then

$$\frac{\text{Density A}}{\text{Density } a}=\frac{\text{mass A}}{\text{mass } a};$$

that is,

$$\frac{D}{d}=\frac{M}{m} \quad . \; . \; . \; (1).$$

If the volumes of the two bodies are unequal, let V and v represent their units of volume respectively; then

$$\text{The mass in a unit of volume of A}=\frac{M}{V},$$

$$\text{"} \qquad \text{"} \qquad \text{"} \qquad a=\frac{m}{v};$$

hence eq. (1) becomes

$$\frac{D}{d}=\frac{\frac{M}{V}}{\frac{m}{v}}=\frac{Mv}{mV} \quad . \; . \; . \; (2).$$

When the masses are equal, or $M=m$, this equation becomes

$$\frac{D}{d}=\frac{v}{V} \quad . \; . \; . \; (3).$$

That is, in this case, the densities are inversely as the volumes; or that which has the greater volume has the less density.

The masses of bodies, *at the same place on the earth*, are simply measured by their weights: thus, if W and w be the respective weights of two bodies whose masses are M and m, then

$$\frac{M}{m}=\frac{W}{w} \quad . \; . \; . \; (4).$$

In this case, eq. (2) becomes

$$\frac{D}{d}=\frac{M\,v}{m\,V}=\frac{W\,v}{w\,V} \quad . \; . \; . \; (5).$$

that is, the densities, or specific gravities, of bodies are directly as their weights, and inversely as their volumes. If the volumes are equal, then this equation becomes

$$\frac{D}{d}=\frac{M}{m}=\frac{W}{w} \quad . \; . \; . \; (6);$$

that is, when the bodies have the same volume, their densities, as well as their masses, are measured by their weights.

Now since the SPECIFIC GRAVITY of a body is its weight as compared with an equal bulk of pure water, the various equations just derived, expressing the ratio of the densities of bodies, will also express the ratio of their specific gravities.

11. DIVISIBILITY. There is no limit to the mathematical conception of the divisibility of space, but the doctrine of the atomic theory seems to indicate that there is a practical limit to the divisibility of matter, that is to say, in going on with our division, we must finally arrive at a certain ultimate particle, or atom of matter, which, from its constitution, no longer admits of separation into parts. Nature presents us with various marvellously minute subdivisions of the particles of matter.

12. COHESION, or the attraction of cohesion, is that property of bodies whereby the atoms composing them are united in a mass. This force of attraction between the particles of matter only takes place at immeasurably minute distances. Bodies are solid, liquid, or aëriform, according as the cohesion of their particles is modified by heat. The particles of gases and vapours repel one another, in consequence of the repulsive force of heat being greater than the force of cohesion; in solids the force of cohesion preponderates over that of repulsion; and in liquids the forces of cohesion and repulsion are presumed to be equal.

13. ELASTICITY is that property of bodies by virtue of which, when their form is altered by the action of an external force, they regain their original form as soon as the external force is withdrawn. All bodies possess this property in a greater or less degree. Most substances have a limit to their elasticity; thus if a straight elastic bar is bent by a pressure applied to it, and if this pressure does not exceed a certain quantity, the bar will resume its original form when this pressure is removed; but on the contrary, if the pressure exceeds a certain quantity, called the limit of

the body's elasticity, the cohesion of the material is injured or destroyed, and then, in this case, the bar will not return to its original form upon the cessation of the pressure. Bodies which have no elastic limit may be called perfectly elastic, such as gases and vapours. Let a portion of air ABDC be confined in a cylinder by means of a piston P exactly fitting the cylinder, so that no air can escape; let pressure be applied to the piston so as to reduce the volume of the air to any space *ab*DC; now when the pressure is removed from the piston the air will resume its former volume ABDC. Liquids scarcely admit of compression, and hence they are called NON-ELASTIC FLUIDS, whereas gases and vapours are called ELASTIC FLUIDS. Some aëriform bodies, such as carbonic acid gas, have been brought to the liquid state by being subjected to high pressure and cold; these are called CONDENSABLE gases: whereas some gaseous bodies, such as oxygen and nitrogen, composing the atmosphere, resist condensation, whatever may be the pressure and cold to which they are subjected; these gases are called PERMANENTLY ELASTIC. Beams, employed in construction, are sometimes considered perfectly elastic, when their resistance to compression, within their limits of elasticity, is equal to their resistance to extension.

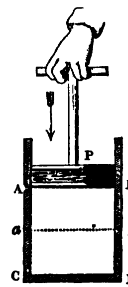


Fig. 1.

14. MOBILITY, or susceptibility to motion, is that property whereby a body admits of change of place. Motion may be ABSOLUTE OR RELATIVE: thus a man in a railway carriage may be in motion *relatively* to the other objects in the carriage, while at the same time he partakes of the absolute motion of the train. In estimating motion there are three things to be considered, viz., THE VELOCITY or quickness of the motion, THE SPACE passed over, and THE TIME in which that space is passed over. The motion of a body is uniform when it passes over equal spaces in equal successive intervals of time; in this case, *the velocity* of the motion is the distance in feet passed over in one second of time, *the space* is the whole distance in feet moved over, and *the time* the number of seconds in which this space is described. In uniform motion, we therefore obviously have

$$\text{the space} = \text{the velocity} \times \text{the time}.$$

If v=the velocity, t=the time, and s=the space, then

$$s = vt \ldots (1).$$

Here there are three general quantities, any two of which being given the remaining one may be found.

15. INERTIA. By this property is meant that matter has no power in itself to change its present state, and that any alteration in its state, whether of rest or motion, must be produced by the action of some external force. If a body is broken, some force must have produced the rupture. If a body is melted, heat must have produced the change. If a body changes its state from rest to motion, some force must have communicated the motion. If it passes from a state of motion to that of rest, some force must have been exerted to destroy the motion. The laws of motion will be hereafter more fully considered.

16. GRAVITY is the tendency which all terrestrial bodies have towards the centre of the earth. When a body is supported, this tendency produces PRESSURE and WEIGHT. The pressure produced by gravity is always exerted in a direction perpendicular to the horizon, and is measured by the weight of the body. The unit of weight in mechanical calculations is a pound, and hence the forces of pressure are usually expressed in units of pounds.

It has been found by experiment (allowance being made for the resistance of the air) that bodies of every size, shape, and weight, fall to the earth exactly in the same manner. Thus, were it not for the resistance of the air, a feather and a guinea would fall from the top of a tower in the same time, and they would strike the ground with the same velocity. *Experiment.*—Take a small piece of thin paper and a penny, and let them fall at the same instant from equal heights above the ground; then the penny will arrive at the ground much sooner than the paper. Here the air presents a greater proportional resistance to the motion of the light body than it does to the heavy one. But in order that the resistance of the air may be the same on both bodies, place the paper on the penny, and then let them fall together: now they both arrive at the ground at the same instant.

Thus it appears that the pressure produced by the earth's attraction upon bodies, is a very different thing from the motion which it generates. In the former case the pressure produced is proportional to the quantity of matter, whereas in the latter case the motion generated, in a given time, is the same for all bodies whatever may be their size, weight, or density. This admits of a satisfactory explanation: the earth attracts every particle of which a body is composed, and hence the weight of a body depends upon the matter which it contains; on the other hand, all the

particles of a body, separated from one another, would evidently fall through the same spaces in the same time; but it appears from experiment, that when the particles are collected in one mass, they fall exactly in the same manner as they would if they were separated from one another.

Gravity is said to take place in consequence of the attraction exerted by the earth upon the body: the general name given to this force is that of the attraction of gravitation. This force is not confined to bodies upon the earth's surface: the moon is maintained in her orbit by the attraction of the earth, and all the planetary bodies in the solar system are subject to the attraction of the sun. The attractive force exerted by bodies on each other is reciprocal and in proportion to their masses; thus, if the body A attracts the body B, then B will also attract A, and the forces which they exert on each other will be proportional to their respective masses. Again, the force of attraction varies inversely as the square of the distance; thus at double the distance the force will be one-fourth, at treble the distance one-ninth, and so on. These two laws are expressed by saying,—that the force of gravitation varies directly as the mass, and inversely as the square of the distance. Bodies are attracted by the earth as if the whole of its mass were collected in its centre; hence the force of gravity at any place depends upon the distance of the place from the centre of the earth. Now since the equatorial diameter of the earth is greater than the polar diameter, it follows that the force of gravity at places near the equator is not so great as it is at places near the poles: thus it is found that a body which produces by its gravity a pound pressure at London would not produce this amount of pressure if taken to the equator; and, in like manner, a pendulum which beats seconds at London, would take a longer time to complete a vibration at the equator.

In consequence of the constant action of the force of gravity, the motion of a falling body becomes quicker and quicker as it descends. In our latitude the velocity acquired by a falling body in one second is $32\frac{1}{6}$ feet, in two seconds it is twice $32\frac{1}{6}$, in three seconds it is three times $32\frac{1}{6}$, and so on. If g be put for $32\frac{1}{6}$ or the velocity acquired in one second, and v for the velocity acquired in t seconds, then

$$v=tg \ldots (1),$$

that is, *the velocity acquired by a falling body increases with the time.*

This law of acquired velocity arises from the fact, that gravity

is a UNIFORM ACCELERATING FORCE, communicating equal increments of velocity in equal times, and that each successive increment of velocity, by the second law of motion, see Art. **24.**, is unaffected by the motion previously acquired. At places towards the equator the accelerating force of gravity is less than it is in our latitude, and at places near the poles it is greater. The laws of descending bodies will hereafter be more fully considered.

MEASURE OF FORCES, &c.

17. When the velocity of a body is continually increased by the action of a force, that force is said to be an ACCELERATING FORCE, and the velocity which it generates in a given time is taken as the measure of the magnitude of the force. Thus if F be put for the force of gravity which generates the velocity g in a second, and F_1 for the force of gravity, at any other place on the earth's surface, which generates the velocity g_1 in the same time, then

$$F : F_1 :: g : g_1 \ldots (1),$$

and in fact if we call g the accelerating force, at the one place, then g_1 will be the accelerating force at the other.

18. The weight of a body, or the pressure which it produces by the force of gravity, at different places on the earth's surface, depends not merely on its mass, but also upon the intensity of gravity. Hence we say, that the weight of a body varies conjointly with the mass and the accelerating force: thus if W be the weight of the body, M its mass, and g the accelerating force, then W varies as the product of M and g, or $W \propto Mg$; and if c represent any constant, we have by algebra,

$$W = cMg \ldots (2),$$

and similarly,

$$W_1 = cM_1 g_1 \ldots (3).$$

In these expressions no value has been assigned to the unit of mass. Now we may give any values we please to W_1, M_1, and g_1 in eq. (3); let them, therefore, be taken each equal to unity, then in this case $c=1$, and eq. (2) then becomes

$$W = Mg \ldots (4),$$

where the unit of M, or the mass, is that quantity of matter which produces a unit of pressure under the action of a unit of accelerating force.

From eq. (4), we have

$$M = \frac{W}{g} \dots (5),$$

that is, the mass of a body is equal to its weight or pressure divided by the accelerating force.

The product of the mass of a body and the accelerating force is called the MOVING FORCE.

19. The relations of weight, mass, and accelerating force of gravity, deduced in the preceding article, are shown by experiment to hold true in reference to any force of pressure giving motion to a body.

Let Q and P be two weights suspended by a cord going over a friction wheel C. Now if the weight Q is greater than P, the former will descend, while the latter will ascend, and their velocities, at any point, will be equal to each other. If we neglect, for the sake of simplicity, the weight of the wheel C and the effect of friction and the resistance of the air, the weight of the mass moved will be $Q+P$, and the pressure giving motion to the mass will be $Q-P$.

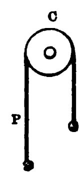


Fig. 2.

Let the velocity generated in the weights, in one second of time, be observed under the following circumstances*:

First.—Let the mass remain the same, that is, let $Q+P$ be constant, but let the relative weights of Q and P be varied, then it will be found that the velocities generated are in proportion to the moving pressure. For example, if v be the velocity generated when the moving pressure or $Q-P=p$, then when the moving pressure or $Q-P=n$ times p, the velocity generated will be n times v.

Hence we conclude, that *the moving pressure varies as the velocity generated*, when the mass is constant.

Second.—Let the moving pressure and the mass be both altered, in such a way as always to have the same proportion to each other, then it will be found that the velocities generated are always equal. For example, if the moving pressure and the weight of the mass moved be respectively $Q-P$ and $Q+P$, or $2(Q-P)$ and $2(Q+P)$, ... or $n(Q-P)$ and $n(Q+P)$, then the velocity generated is always the same.

* These experiments are best performed by an apparatus contrived by Atwood.

Hence we conclude, *that the moving pressure varies as the mass moved*, when the velocity generated is constant.

Now when both the velocity and mass change, we have, by the rules of algebra,

the moving pressure varies as the product of the mass and the velocity generated.

But as the velocity generated is the measure of the accelerating force, we also have

the moving pressure varies as the product of the mass and the accelerating force.

And moreover, from the definition of the moving force, we also have

the moving pressure varies as the moving force.

20. FORCES, whether impulsive or constant, which communicate motion to a body, are measured by the momentum produced in the body, that is, by the velocity multiplied into the mass of the body moved.

It is found by experiment, that if two inelastic bodies directly impinge upon each other, with velocities which are inversely as their masses, then they will exactly destroy each other's velocity. This shows, that when the bodies have equal momenta, the forces of pressure, resulting from their momenta, are exactly equal to each other. For example, if a body weighing 3lbs. has a velocity of 6ft., and another body weighing 2lbs. has a velocity of 9ft., then their quantities of motion or momenta are equal; and if the bodies were to come into direct collision with each other, they would exactly destroy each other's motion, supposing them to be perfectly inelastic, and the impulsive forces which communicated motion to them would be equal.

21. FORCES OF PRESSURE, or statical forces, generally are represented by straight lines of definite lengths, the number of units of length in the line being taken equal to the number of units of lbs. in the pressure; the direction of the line also represents the direction of the pressure; and the commencement of the line, the point at which the pressure is applied. A pressure is given when its magnitude, direction, and point of application are given.

When a weight is suspended by a cord, this weight produces in the cord A FORCE OF TENSION, which is measured by the weight. Rods and cords are used for the transmission of pressure: in all

mechanical investigations, unless the contrary is stated, the rods are supposed to be inflexible and without weight, and the cords to be perfectly flexible and without weight; and in all such cases the force of pressure is supposed to be transmitted without loss.

The expansibility of condensed air or steam produces a FORCE OF PRESSURE, which is estimated by the number of pounds exerted on one inch of surface.

When a horse draws a load, the force which he exerts is called THE FORCE OF TRACTION, which is usually estimated in units of pounds.

22. When forces tend to destroy motion, they are called forces of resistance, or simply resistances, such as the resistance of the air, the resistance of friction, &c.

23. FRICTION. When a body is slowly moved along a horizontal plane, the resistance to be overcome is due to friction; this resistance must obviously be greater or less according to the degree of roughness or smoothness of the rubbing surfaces. The following laws of friction have been discovered by experiment:

First. — The resistance of friction, on a given surface, is always a certain proportional part of the weight of the body.

Second. — The resistance of friction is not affected by the extent of the rubbing surfaces.

Third. — The resistance of friction is not affected by any change in the rate of the body's motion.

Let W be a weight drawn upon the horizontal plane HR, by a force, acting parallel to the plane, produced by a weight P suspended from a cord passing over the wheel C; then the weight P, just necessary to draw W along the plane, will be equal to the resistance of friction.

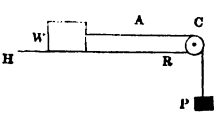


Fig. 3.

Now from the first law of friction, P will always be a certain proportional part of W for the same rubbing surfaces; let f represent the fraction giving this proportional part, then we have

$$P = f W \ldots (1),$$

or putting F for the resistance of friction, then as $F = P$, we have

$$F = f W \ldots (2).$$

Here the constant f is called THE COEFFICIENT OF FRICTION; its value depends upon the nature of the rubbing surfaces.

In these expressions, there are three quantities, any two of

which being given, the remaining one may be found; thus we have

$$W = \frac{F}{f} \ldots (3),$$

$$\text{and } f = \frac{F}{W} \ldots (4).$$

When the weight is expressed in units of tons, it is often convenient to give the coefficient of friction in units of lbs. for every unit of tons in the weight. In this case, let T = the units of tons in the weight, p=the number of lbs. resistance of friction for one ton, then

$$F = pT \ldots (5).$$

By the second and third laws of friction, these various relations, just derived, subsist whatever may be the extent of the rubbing surfaces, or the velocity with which the weight is moved.*

In these expressions P or F will be the horizontal traction acting parallel to the plane, just necessary to draw the weight along the plane.

Example 1. What must be the horizontal traction to draw a weight of 1 ton along a level road whose coefficient of friction is $\frac{1}{30}$?

Here $W = 2240$ lbs., $f = \frac{1}{30}$, therefore by eq. (1),

$$P = \frac{1}{30} \times 2240 = 74\tfrac{2}{3} \text{ lbs.}$$

Example 2. A horizontal traction of 80 lbs. just draws a weight of 1 ton along a level road; required the coefficient of friction.

Here by eq. (4), we have

$$f = \frac{80}{2240} = \frac{1}{28}.$$

LAWS OF MOTION.

24. The truth of the three following laws of motion is based upon observation and experiment:—

FIRST LAW OF MOTION.—*A body in motion will move continually in a straight line and with a uniform velocity, if it is not acted on by any external force.*

* It should be observed that the coefficient of friction for bodies bordering on a state of motion, is a little different from that which has a relation to bodies actually in motion.

For the proof of this law, the student may consult any popular work on Mechanics.

SECOND LAW OF MOTION.—*If any number of forces act at the same instant upon a body in motion, each force produces its full effect in the direction of its action, just as if it had acted alone upon the body at rest.*

Thus, if a ball be dropped from the top of the mast of a ship moving uniformly, the ball strikes the deck at the bottom of the mast, and falls precisely in the same time as if the ship were at rest.

Although the earth, by its diurnal motion, carries all bodies on its surface uniformly from west to east, yet all motions take place on the earth's surface just as if it were at rest.

If a ball be thrown along the deck of a vessel, moving uniformly, it will move on the deck in precisely the same manner as if the vessel were at rest. Let S represent the deck of the vessel moving uniformly in the water; suppose the vessel to move from S to *s*, or that the point A moves from A to C, in the same time that the ball moves from A to B. Now whilst the ball is moving on the line AB across the deck, it is at the same time carried with the vessel from A to C, and at the end of the time the ball is found at D; so that it preserves its two motions; that is to say, it moves in the direction AB as if it had no other motion, and in the direction AC with the vessel as if it had no motion on the deck. The actual path pursued by the ball is evidently in the diagonal AD of the parallelogram ABDC.

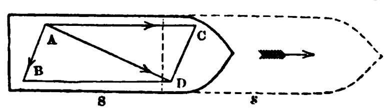


Fig. 4.

This establishes what is called the parallelogram of motion, which may be enunciated as follows:—

PARALLELOGRAM OF MOTION.—*If two velocities be given to a body at the same instant, the actual velocity will be represented by the diagonal of the parallelogram formed upon the two lines representing the velocities impressed upn the body.*

Let a body at A have a velocity given to it which would cause it to move uniformly from A to C in a given time, and another velocity at the same instant, which would cause it to move uniformly from A to B in the same time: now if the parallelogram ABCD be completed, the actual path of the body will be the diagonal AD described in the same time.

THIRD LAW OF MOTION. *When a pressure acting on a body*

puts it in motion, the momentum generated in a unit of time is proportional to the pressure.

This law is proved in Arts. **19.** and **20.**

An important consequence of this law of motion is *the principle of action and reaction which are always equal and contrary.*

Exercises.

Densities.—Art. 10.

1. Two bodies, each of uniform density, are of the same size, and weigh 4 lbs. and 5 lbs. respectively, at the same place on the earth. Compare their densities. *Ans.* 1 to 1·25.

2. A substance containing 7 c. in. weighs 3 oz.; and another substance containing 6 c. in. weighs 14 oz.; compare their densities.

Here by eq. (5), Art. **10.**, we have

$$\frac{\text{D}}{d}=\frac{14\times7}{3\times6}=5\tfrac{4}{9};$$

that is to say, the one body is $5\frac{4}{9}$ times the density of the other, or their densities are in the ratio of 49 : 9.

3. Required the same as in the last example when the substances contain 5 and 6 c. in. respectively, and weigh 10 and 18 lbs. *Ans.* 3 : 2.

4. A cubic foot of a substance weighs w lbs., and another substance having twice the density weighs W lbs., required its magnitude. *Ans.* $\frac{\text{W}}{2w}$ cubic feet.

Velocities.—Arts. 14. 16.

5. A railway train moves at the rate of 30 miles an hour; what is its velocity in feet per second? *Ans.* 44.

6. A body moves with the uniform velocity v. With what velocity must another body move which starts from the same point a hours after the former, and overtakes it in b hours? *Ans.* $\frac{v(a+b)}{b}$.

7. A body moves with a velocity of v feet for t seconds, and for the next t_1 seconds with the velocity of v_1 feet; required the *mean velocity* of the body, or that uniform velocity which would have carried it over the same space in the same time. *Ans.* $\frac{vt+v_1t_1}{t+t_1}$.

8. What velocity would the force of gravity give to a falling body in 5 seconds? *Ans.* $160\frac{5}{6}$ ft.

9. In what time would the force of gravity communicate a velocity of 193 feet? *Ans.* 6 sec.

10. The velocity which a falling body would acquire in 2 seconds, is one-third the velocity which it would acquire in 6 seconds. If v be the velocity acquired in t seconds, and v_1 the velocity acquired in t_1 seconds, then

$$v : v_1 :: t : t_1$$

that is to say, the velocities acquired are proportional to the times.

Measure of Forces.—Arts. **17, 18, 19. 23, 24.**

11. What velocity would gravity communicate to a falling body in one second, if its force were one-fourth of what it really is in our latitude? *Ans.* $8\frac{1}{24}$ ft.

12. At a certain point above the earth's surface a weight of w lbs. (estimated by a spring weighing machine) only weighed w_1 lbs.; what velocity would a falling body there acquire in one second? (See Arts. **17.** and **19.**) *Ans.* $\frac{w_1}{w}.g$.

13. At the same place on the earth the mass of a body is proportional to its weight.

14. If a weight of 4 lbs. have a velocity of 5 ft., what must be the weight of another body having a velocity of 2 ft., so that its momentum may be equal to the former? *Ans.* 10 lbs.

15. What force will be required to draw a load of 10 cwts. along a horizonal road whose coefficient of friction is $\frac{1}{25}$?

Ans. $\frac{2}{5}$ cwts.

16. If a pressure of 2 cwts. be required to move a load of 40 cwts. along a rough horizontal plane, required its coefficient of friction. *Ans.* $\frac{1}{20}$.

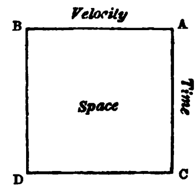
17. If a ball be placed at one of the corners of a smooth table, how should the ball be struck so that it may move towards the opposite corner of the table?

ON ACCELERATING FORCES AND MOTION.

Graphical representation of the space described by a moving body.

25. It has been shown, Art. **14.** that the space passed over by a body, moving with a uniform velocity, is equal to the product of

the time by the velocity. Now if a rectangle be constructed, having the units in the side AC equal to the units of time of the body's motion, and the units in the side AB equal to the units of velocity; then the units of surface in the rectangle will represent the space passed over by the body. In like manner, the space described by a body with a *variable* velocity, may be graphically represented. Let the units in AQ (see *fig.* 7. p. 24.) represent the units of time of a body's motion, AC, CE, . . . , SQ very small successive intervals of time, and AB, CD, . . . , QR the velocities which the body has at these corresponding intervals of time; then supposing a line to be drawn through the extremities of these ordinates, the space described by the body will be represented by the area of the curved space ABRQ. For suppose the body to move through the interval AC with a certain velocity, which is the mean between the velocity at A, and the velocity at C, then the space described will be represented by the area of the trapezoid ABDC; in like manner, the area of the trapezoid CDFE will represent the space described by the body with a velocity which is the mean between CD and EF; and so on: therefore the whole area ABRQ will represent the whole space described by the body in the whole time AQ, each interval being described with the mean velocity of that interval. Now the less the intervals are made the nearer and nearer does the mean velocity of each interval approach the actual velocity of that interval, and therefore, on the principle of ultimate ratios, the curvilineal area ABRQ will truly represent the space described.



To find the relations of the space, time, and velocity of a body acted freely on by the force of gravity.

26. We have, from eq. (1), Art. **16.**, for the relation of time and velocity acquired

$$v=tg \ldots (1).$$

In order to determine the relation of time and space; let AB (of the right angled triangle ABC) represent the units of time of a body's descent in falling from a state of rest, and BC the units of velocity acquired in that time; from any point D draw DE perpendicular to AB, then DE will represent the velocity acquired in falling during the units of time represented by AD. For by eq. (1) the velocity acquired is proportional to the time; but by the similar

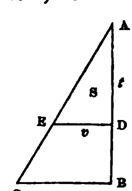


Fig. 5.

triangles ABC and ADE, the perpendicular DE bears the same proportion to AD that BC does AB, therefore DE will represent the velocity acquired in falling for the units of time represented by AD. And the same thing will hold true for any other perpendicular. Hence it follows (Art. 25.) that the space described by a body in falling from a state of rest is represented by the area of the triangle ABC, when the units in AB represent the time, and the units in BC the velocity acquired in that time.

Put $t=$AB, $v=$BC, and s=the space described, then

$$\text{area ABC} = \tfrac{1}{2}\,\text{BC} \times \text{AB}$$
$$= \tfrac{1}{2}\,vt,$$
$$\therefore \text{s} = \tfrac{1}{2}\,vt \ldots (2).$$

Here $\frac{1}{2}v$ is the mean velocity, and therefore THE SPACE DESCRIBED IS EQUAL TO THE PRODUCT OF THE MEAN VELOCITY BY THE TIME.

Substituting the value of v given in eq. (1), we get

$$\text{s} = t^2 \times \frac{g}{2} \ldots (3).$$

If the body be projected vertically downwards with the velocity V, then

$$\text{the space due to projection} = t\text{V};$$

but we have from the second law of motion,

$$\text{s} = \text{space due to projection} + \text{space due to gravity},$$
$$= t\text{V} + t^2 \times \frac{g}{2} \ldots (4).$$

To find the space, s, through which a body must fall in order to acquire a given velocity v.

By eq. (1), we have

$$\text{the time of descent, or } t = \frac{v}{g},$$

and substituting this value of t in eq. (3), we have

$$\text{s} = \left(\frac{v}{g}\right)^2 \times \frac{g}{2}$$
$$= \frac{v^2}{2g} \ldots (5).$$

If the space be given to find the velocity acquired, then we have from this equality,

$$v = \sqrt{2g\text{s}} \ldots (6).$$

If a body be projected vertically upwards with the velocity v, it is evident that it will rise to the same height as that through

which it must fall in order to acquire the proposed velocity; hence eq. (5) also expresses the height to which a body will rise when projected upwards with the given velocity v.

If AD be the space through which a body must fall to acquire the velocity v, and KD the space to acquire the velocity v_1; then by eq. (1),

$$\text{AD}=\frac{v^2}{2g}, \text{ and } \text{KD}=\frac{v_1^2}{2g},$$

hence we get by subtraction,

$$\text{AK}=\frac{v^2-v_1^2}{2g} \ldots (7),$$

Fig. 6.

which is the expression for the space through which a body must fall in order to change its velocity from v_1 to v.

To find the space s_1, through which a body will fall during the last t_1 seconds of its motion.

Here we have

$$s=t^2\times\frac{g}{2},$$

$$\text{and } s-s_1=(t-t_1)^2\times\frac{g}{2},$$

$$\therefore\ s_1=\{t^2-(t-t_1)^2\}\frac{g}{2} \ldots (8).$$

If a body be projected vertically downwards with the velocity V, what will be its velocity (v) after descending s feet?

By eq. (2),

$$\text{velocity acquired by gravity}=\sqrt{2gs}$$

$$\therefore\ v=V+\sqrt{2gs} \ldots (9).$$

Exercises for the Student.

1. A person while ascending in a balloon with a vertical velocity of V feet per second, lets fall a stone when he is h feet above the earth; required the time in which the stone will reach the ground.

Ans. $\dfrac{V+\sqrt{V^2+2gh}}{g}$.

2. Show that the space described by a body during the nth second of its fall is equal to $(2n-1)\dfrac{g}{2}$; and consequently that the spaces described during successive equal intervals of time, are as the odd numbers 1, 3, 5, 7, &c.

3. A body falling from a state of rest describes s feet during the the last second of its fall; required the total time of its fall.

$$Ans. \frac{2s+g}{2g}.$$

4. A body A is projected vertically downwards from the top of a tower with the velocity V, and one second afterwards another body B is let fall from a window a feet from the top of the tower; in what time, t, will A overtake B?

$$\text{Space moved over by } A = Vt + t^2 \times \frac{g}{2},$$

$$\text{and } ,, \quad ,, \quad ,, \text{ by } B = (t-1)^2 \times \frac{g}{2}.$$

Now the differences of these spaces must be equal to a;

$$\therefore\ Vt + t^2 \times \frac{g}{2} - (t-1)^2 \times \frac{g}{2} = a;$$

$$\therefore\ t = \frac{2a+g}{2(V+g)}.$$

5. A body is thrown vertically upwards with the velocity of v feet per second; required the height to which it will ascend in t_1 seconds. *Ans.* $vt_1 - t_1^2 \times \frac{g}{2}$.

6. A body is thrown vertically upwards with the velocity of v feet per second; at the same instant another body is let fall from the height of h feet above the ground: in what time will the bodies pass each other? *Ans.* $\frac{h}{v}$.

7. A stone let fall into a well, is heard to strike the water in t seconds; required the depth of the well, supposing the velocity of sound to be a feet per second. *Ans.* $\left\{\sqrt{at + \frac{a^2}{2g}} - \frac{a}{\sqrt{2g}}\right\}^2$.

8. A body is thrown vertically upwards with a velocity of v feet per second; required the time when it is at a given height h.

Ans. $\frac{v \pm \sqrt{v^2 - 2gh}}{g}$, where the lower sign expresses the time of the body's ascent, and the upper sign the time of its returning in its descent.

To find the relations of the space, time, and velocity of a body acted on by any constant moving pressure P.

27. Let w = the weight of the body, and g_1 = the accelerating force.

In the foregoing formulæ, g is put for the accelerating force of

gravity acting freely upon the body; that is to say, g is the velocity, which gravity communicates to any body during one second of its fall. In the present case, w, the weight of the body or mass of matter, is the moving pressure which generates the velocity g in one second; and p is the moving pressure, acting on the same mass of matter, which generates the velocity g_1 in one second; hence we have, by Art. **19.**,

$$w : p :: g : g_1,$$

$$\therefore g_1 = \frac{p}{w} \cdot g \ldots (1);$$

that is to say, *the accelerating force is equal to the moving pressure acting on the body divided by its weight and multiplied by the accelerating force of gravity.*

All the foregoing relations will hold true when g_1 is substituted for g.

Thus if v_1 be put for the velocity which the body has at the commencement of the time t, then we have from eqs. (1) and (4), Art. **26.**,

$$v = v_1 \pm g_1 t \ldots (2),$$

$$s = t v_1 \pm t^2 \times \frac{g_1}{2} \ldots (3),$$

where the + or − signs are taken, according as the force accelerates or retards the motion of the body.

From eq. (2) we get

$$t = \pm \frac{v - v_1}{g_1} \ldots (4),$$

and substituting this value of t in eq. (3), we get

$$s = \pm \frac{v^2 - v_1^2}{2 g_1} \ldots (5);$$

where the value of g_1 is given in eq. (1); by substituting this value in eqs. (2), (3), and (5), we obtain

$$v = v_1 \pm \frac{pg}{w} \cdot t \ldots (6),$$

$$s = t\, v_1 \pm t^2 \times \frac{pg}{2w} \ldots (7),$$

$$s = \pm \frac{w(v^2 - v_1^2)}{2g \cdot p} \ldots (8),$$

From this last equality, we have

$$s \cdot p = \pm \frac{w(v^2 - v_1^2)}{2g} \ldots (9);$$

but $s \cdot p$ is the work of the moving pressure p,

$$\therefore \text{ Work of the moving pressure, } u = \pm \frac{w(v^2 - v_1^2)}{2g} \ldots (10).$$

CHAP. II.

PRELIMINARY PROPOSITIONS AND PROBLEMS RELATIVE TO LABOURING FORCES.

28. WHENEVER there is a resistance moved over a certain space, there must be labour or work done; and it is obvious that the amount of labour or work must vary with the resistance moved, and the space through which it is moved. Labouring force is exerted, and therefore *work* is done, in sawing, grinding, dragging, raising heavy weights, &c. In order to compare labouring forces with one another, it is necessary that we should fix upon some unit of work or labour. Now since every unit of measure must be of the same kind as the thing to be measured, the following definition has been adopted in this country:— A UNIT OF WORK is the labour expended in moving a pressure of one pound through the space of one foot in a direction contrary to that in which the pressure acts. Thus, for example, a unit of work is performed when a pound weight is raised one foot in opposition to gravity; or if the body W, moving along the horizontal plane H R (see Art. **23.**, *fig.* 3.), produces a friction of 1 lb., then a unit of work is performed when the body has been moved on the plane over the space of one foot.

UNITS OF WORK, OR WORK.

29. *The work expended in raising a body is equal to the product of its weight in lbs. by the vertical space in feet through which it is raised.*

When the weight is constant, the work must obviously be proportional to the distance, and when the space is constant, the work must be proportional to the weight. Let P = the weight of the body in lbs., S = the vertical space in feet through which this weight is moved, and U = the units of work done; then

No. units of work in raising 1 lb. over 1 ft. $=1$,

∴ „ „ „ 1 lb. over s ft. $=s$,

∴ „ „ „ W lbs. over s ft. $=P\times S$;

that is, $U=P\times S\ldots(1)$.

We may obviously in general consider U, in this formula, as the work expended in moving a pressure or resistance of P lbs. through a space of S feet, without regard to the direction in which that pressure is exerted. Taking this more enlarged view of the formula, we conclude that *the work is equal to the product of the resistance in lbs. by the space in feet through which the resistance is moved.*

30. If we take the units in the two adjacent sides of a rectangle to represent the units of pressure and space respectively, then the units of work will be equal to the units of surface in the rectangle.

31. UNITS OF HORSE-POWERS.—Watt estimated that a horse could perform 33,000 units of work per minute; this work is therefore adopted as the measure of a horse-power.

If U be the work performed by an engine per minute, and N the number of its horse-powers, then

$$N=\frac{U}{33000}\ldots(1).$$

32. MODULUS OF AN ENGINE.—The modulus of a machine is that fraction which expresses the relation of the work done to the work applied. Owing to the differences of construction in machines, the modulus of one machine frequently differs very much from that of another. The modulus of a machine, therefore, expresses its relative efficiency.

Let $U=$ the work applied to a machine, $U_1=$ the work yielded, or the useful work, and $m=$ the modulus of the particular machine; then

$$U_1=mU\ldots(1).$$

In a perfect machine, where there is no loss of work by transmission through the parts, the modulus is unity; for in this case the work yielded by the machine would be exactly equal to the work applied.

From eq. (1), we get

$$U=\frac{U_1}{M}\ldots(2);$$

$$M=\frac{U_1}{U}\ldots(3).$$

Eq. (1) expresses the useful work in terms of the work applied;

eq. (2) is the converse of (1); and eq. (3) shows that the modulus is the fraction obtained by dividing the work done by the work applied.

It will be afterwards shown that there are exact modes of expressing the relation of U to U_1.

WORK OF A VARIABLE PRESSURE.

33. When the pressure exerted through a given space varies, the work done may be determined by multiplying the given space by the mean of all the variable pressures.

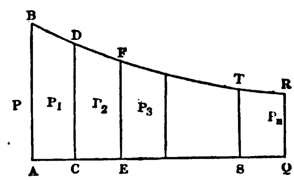


Fig. 7.

Let AQ represent the space in units of feet through which a variable pressure is exerted; let AQ be divided into n equal parts, AC$=$CE$=$&c.$=$SQ; and suppose P_0, P_1, P_2, . . ., P_n, to be the pressures applied at the points A, C, E, . . ., Q, respectively; take the units in the perpendiculars AB, CD, EF, . . ., QR, equal to the units of lbs. in the pressures P, P_1, P_2, . . ., P_n, respectively; then the work done from A to C will be equal to the mean of the pressures P and P_1, multiplied by the space AC, that is, the work will be equal to the units of surface in the area ACDB; in like manner the work done from C to E will be equal to the area CDFE; and so on; so that the work done through the whole space AQ by a pressure which continuously varies in the manner described, will be equal to the units of surface in the area ABRQ. This area may be found approximately by the ordinary rule of Mensuration for the area of a curved space having equidistant ordinates, or more accurately by Thomas Simpson's rule (see the Author's "Geometry and Mensuration," pp. 160. and 185.).

Let u=the work done from A to Q; S=the space AQ; and a= AC=&c., the common distance between the ordinates; then we have by the ordinary rule,

$$u = \text{area ABRQ}$$

$$= \frac{S}{n}(\tfrac{1}{2}P + P_1 + P_2 + \ldots + P_{n-1} + \tfrac{1}{2}P_n) \ldots (1);$$

or by Simpson's rule, the number of spaces being even,

$$u = \text{area ABRQ}$$

$$= \frac{S}{3n}\{P + P_n + 4(P_1 + P_3 + \ldots P_{n-1}) + 2(P_2 + P_4 + \ldots + P_{n-2})\} \ldots (2).$$

If P' be put for the mean pressure, then

$$P' \times S = u, \text{ and } P' = \frac{u}{S} \ldots (3).$$

The following applications of the principles of WORK will form a useful introduction to the subject of industrial mechanics.

WORK IN PUMPING WATER AND RAISING WEIGHTS.

34. *Problems.*

(1.) How many units of work would be required to raise A cubic feet of water to the height of a fathoms?

$$\text{Weight water in lbs.} = 62 \cdot 5A,$$
$$\text{Height raised in feet} = 6a,$$

$$\begin{aligned} \therefore \text{ Work} &= \text{weight water in lbs.} \times \text{height raised in ft.} \\ &= 62 \cdot 5A \times 6a \\ &= 375Aa. \end{aligned}$$

(2.) If a horse draw w cwts. out of a well or pit, by means of a cord going over a wheel, moving at the rate of m miles per hour; how much work, U, will he perform per minute?

$$\text{Weight in lbs.} = 212 \times w,$$

$$\text{Space in feet moved over per min.} = \frac{5280\,m}{60} = 88\,m;$$

$$\begin{aligned} \therefore \text{ Work per min., or } U &= 212 \times w \times 88m \\ &= 18656\,m\,w. \end{aligned}$$

If U and m are given and w is required, we have, from this equality,

$$w = \frac{U}{18656m}.$$

3. How many horse powers, H. P., would it take to raise T tons of coals per hour from a pit whose depth is a fathoms?

$$\text{Work per hour} = T \times 2240 \times a \times 6;$$

$$\therefore \text{ Work per min.} = \frac{T \times 2240 \times a \times 6}{60} = 224\,a\,T;$$

$$\therefore \text{ H. P.} = \frac{224\,a\,T}{33000}.$$

In this equality there are three general quantities, viz., a, T, and H. P., any two of which being given the remaining one may be found.

4. Required the work in raising water from three different levels, whose depths are a, b, c fathoms respectively; from the first A cubic feet of water are to be raised per min., from the second B cubic feet per min., and from the third C cubic feet per min.

$$\text{Work in raising the water from the first level} = 62{\cdot}5\,\text{A} \times a \times 6$$
$$= 375 \times a \,.\, \text{A};$$

and so on for the work in the other levels;

$$\therefore \text{ Work per min. or } \text{U}_1 = 375\,(a\,.\,\text{A} + b\,.\,\text{B} + c\,.\,\text{C}).$$

5. If M be put for the modulus of the pumps in the last problem, what must be the horse powers, H. P., of the engine, in order to pump all the water.

Let U_1 = the useful work determined in the last problem; and U = the work of the engine per min.; then we have, from eq. (2) Art. 32.,

$$\text{U} = \frac{\text{U}_1}{\text{M}},$$

$$\therefore \text{ H. P.} = \frac{375}{33000\,\text{M}}\,(a\,.\,\text{A} + b\,.\,\text{B} + c\,.\,\text{C}).$$

6. If U be the work applied to a machine, and M its modulus, to find an expression for the work lost.

$$\text{Work lost} = \text{U} - \text{U}_1$$
$$= \text{U} - \text{M}\,.\,\text{U} = \text{U}\,(1 - \text{M}).$$

7. Required the number of horse powers, N, to raise T tons of coals per hour from a pit whose depth is a fathoms, and at the same time to give motion to a forge hammer whose weight is w lbs., which makes n lifts per minute of h feet each.

$$\text{Work in raising coals per min.} = 224\,a\,\text{T},$$
$$\text{,, \quad ,, \quad hammer \quad ,,} \quad = nhw;$$
$$\therefore \text{N} = \frac{224\,a\,\text{T} + n\,h\,w}{33000}.$$

In this equality there are six general quantities, any five of which being given, the remaining one may be found. Thus let T be required, then

$$\text{T} = \frac{33000\,\text{N} - n\,h\,w}{224\,a}.$$

If w be required, then

$$w = \frac{33000\,\text{N} - 224\,a\,\text{T}}{n\,h}.$$

8. If a man working with a certain machine can perform u

effective units of work per minute, how many lifts, n, will he give per hour, to a pile ram weighing w lbs., and having a lift of h feet.

$$\text{Work done by the man per hour} = 60u,$$

$$\text{Work in raising ram per hour} = whn,$$

$$\therefore\ whn = 60u,$$

$$\therefore\ n = \frac{60u}{wh}.$$

9. An engine of N effective horse powers is found to pump A cubic feet of water, per minute, from a depth of a fathoms; required an expression for the modulus of the pumps.

$$\text{Work engine per min.} = N \times 33000,$$

Useful work or work expended in pumping water $= 62{\cdot}5\,A \times 6a$;

hence we have by eq. (3) Art. **32.**,

$$M = \frac{U_1}{U} = \frac{62{\cdot}5\,A \times 6a}{N \times 33000} = \frac{A\,.\,a}{88\,N}.$$

10. There were A cubic feet of water in a mine whose depth is a fathoms, when an engine of N horse powers began to work the pump; now the water continued to flow into the mine at the rate of A_1 cubic feet per minute; required the time in which the mine would be cleared of water, allowing M to be the modulus of the pump.

Let x = the no. minutes to clear the mine of water.

$$\text{Weight water to be pumped} = 62{\cdot}5\,(A + A_1 x),$$

$$\therefore\ \text{Work in pumping water} = 375a\,(A + A_1 x),$$

$$\text{Effective work engine} = M \times N \times 33000 \times x,$$

$$\therefore\ M \times N \times 33000 \times x = 375a(A + A_1 x),$$

$$\therefore\ x = \frac{a\,.\,A}{88\,.\,M\,.\,N - a\,.\,A_1}.$$

11. What must be the horse powers, N, of an engine working for e hours per day, to supply n families with g gallons of water each per day, supposing the water to be raised to the mean height of h feet, and that a gallon of water weighs 10 lbs?

$$\text{Work in pumping water per day} = 10ngh;$$

$$\text{Effective work engine per day} = M \times N \times 33000 \times 60 \times e;$$

$$\therefore\ M \times N \times 33000 \times 60 \times e = 10ngh,$$

$$\therefore\ N = \frac{ngh}{198000\,eM}.$$

If $e=10$, $n=3000$, $g=80$, $h=70$, and $M=\frac{2}{3}$; then this formula becomes

$$N=\frac{3000\times 80\times 70}{198000\times 10\times \frac{2}{3}}=12\cdot 7.$$

WORK IN OVERCOMING THE RESISTANCE OF FRICTION.

35. If U be put for the work due to friction in moving a body over S feet on a horizontal plane; then we get from eqs. (2) and (5), Art. **23.**,

$$U=fWS \ . \ . \ . \ (1)$$

$$\text{or } U=pTS \ . \ . \ . \ (2)$$

If P be put for the traction requisite to draw the weight along the horizontal plane, then from eq. (1) Art. **23.**, we have

$$P=fW \ . \ . \ . \ (3).$$

When a body (such as a railway train) moves on a plane with a uniform velocity, the work done upon the body to sustain that velocity is equal to the work requisite to overcome the resistances; and this speed is said to be the maximum speed, because it is the greatest which the body can have with the given moving pressure.

Problems.

1. Required the work performed per minute, U, in sustaining a body weighing W lbs. at a uniform rate of m miles per hour on a horizontal plane, whose coefficient of friction is f,

$$\text{Resistance of friction in lbs.}=fW,$$

$$\text{Space in feet moved over per min.}=\frac{m\times 5280}{60}=88m,$$

$$\therefore\ U=fW\times 88m=88fmW.$$

Here there are four general quantities, any three of which being given, the remaining ones may be found. Thus if W be required, we have

$$W=\frac{U}{88fm}.$$

2. What must be the horse powers, N, of a locomotive engine, which moves with the uniform or maximum speed of m miles per hour on a level rail, the weight of the train being T tons, and the friction p lbs. per ton, all other resistances being neglected?

Work in overcoming the resistance per min.

$$=p\,\text{T}\times\frac{5280\,m}{60}=88\,mp\,\text{T};$$

$$\therefore\ \text{N}=\frac{88\,mp\,\text{T}}{33000}=\frac{8\,mp\,\text{T}}{3000}\ \ldots\ (1).$$

If T in this equality be required, the other general quantities being given, we get

$$\text{T}=\frac{3000\,\text{N}}{8\,mp}\ \ldots\ (2).$$

If m be required, we get

$$m=\frac{3000\,\text{N}}{8\,p\,\text{T}}\ \ldots\ (3).$$

Example 1. Let $m=30$, $\text{T}=50$, $p=8$, then from eq. (1), we have

$$\text{N}=\frac{8\times30\times8\times50}{3000}=32.$$

Example 2. Let $\text{N}=40$, $m=35$, and $p=8$; required T.
Here by eq. (2), we have

$$\text{T}=\frac{3000\times40}{8\times35\times8}=53{\cdot}5 \text{ tons}.$$

Example 3. Let $\text{T}=80$, $\text{N}=70$, and $p=8$; required m.
Here by eq. (3), we have

$$m=\frac{3000\times70}{8\times8\times80}=41 \text{ miles}.$$

3. If $t=250-41\frac{2}{3}\,r$, express the relation between t, the traction of a horse in lbs., and r the rate in miles per hour, what gross weight W will the horse draw on a plane whose coefficient of friction is f.

Here by eq. (3) we have

$$t=f\text{W},$$

$$\therefore\ f\text{W}=250-41\tfrac{2}{3}\,r,$$

$$\therefore\ \text{W}=\frac{250-41\frac{2}{3}\,r}{f}\ \ldots\ (1).$$

If the rate be required, the weight and the coefficient of friction being given, then

$$r=\frac{250-f\text{W}}{41\frac{2}{3}}\ \ldots\ (2).$$

To determine the work, U, done per min., we have

$$\text{U} = \text{traction in lbs.} \times \text{space in feet per min.}$$
$$= (250 - 41\tfrac{2}{3}r) \times 88r \ldots (3).$$

Here the work done depends solely upon the speed.

Example 1. Let $r=3$, and $f=\frac{1}{12}$, to find W.

Here by eq. (1.), we get

$$\text{W} = \frac{250 - 41\frac{2}{3} \times 3}{\frac{1}{12}} = 1500 \text{ lbs.}$$

Example 2. Let $\text{W}=2240$, and $f=\frac{1}{30}$, required r.

Here by eq. (2), we get

$$r = \frac{250 - \frac{1}{30} \times 2240}{41\frac{2}{3}} = \frac{750 - 224}{125} = 4{\cdot}2 \text{ miles.}$$

Example 3. If the speed of the horse be 3 miles per hour what work will he perform per minute.

Here $r=3$, therefore by eq. (3.) we have

$$\text{U} = (250 - 41\tfrac{2}{3} \times 3) \times 88 \times 3 = 33000.$$

If $r=2$, then

$$\text{U} = (250 - 41\tfrac{2}{3} \times 2) \times 88 \times 2 = 29333.$$

If $r=4$, then $\text{U}=29333$.

It results from formula (3), that a horse performs the greatest amount of work when he travels at the rate of 3 miles per hour. With this speed he performs a standard unit of horse power.

4. What must be the horse powers of an engine to cut A sq. ft. of planking in a day of n hours long, allowing u to be the units of work requisite to saw a square foot of the same timber.

$$\text{Work in cutting 1 sq. ft.} = u,$$
$$\therefore \text{ Work in cutting A sq. ft.} = \text{A}u,$$
$$\therefore \text{ Work per min.} = \frac{\text{A}u}{60\,n},$$
$$\therefore \text{ H. P.} = \frac{\text{A}u}{60 \times 33000 \times n}.$$

Work in moving a Body on an Inclined Plane.

36. The work performed in moving a body up an inclined plane, without friction, is equal to the product of the weight of the body in lbs. by the vertical height in feet through which it is raised. This principle will be hereafter established on strictly mathematical reasoning; but the following simple method of proof is well deserving attention.

Let ABC be an inclined plane, having the base AB horizontal and the side BC vertical. Suppose a uniform and perfectly flexible chain ACBD to be thrown round the plane as shown in the annexed cut. The chain can have no tendency to motion: for whatever

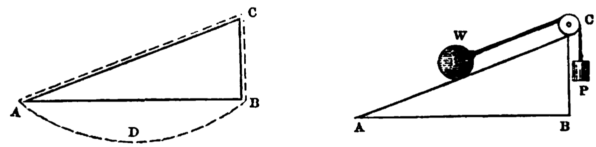


Fig. 8.

may be its position it will always have the same form and relative position of parts, and therefore if the force of gravity be supposed to give motion to it in any one position, the same cause would continue to act in every other position, and consequently the chain would move on for ever, which is impossible.

The curve ADB which the chain assumes, by the action of gravity, is symmetrical on each side of D, and the tension at A must be the same as the tension at B. The portion ADB may therefore be removed and the remaining portion ACB will still remain at rest; and then the tendency of the portion AC to move down the plane will be balanced by the gravity of the vertical portion BC. Let W be the weight of the portion AC, and P that of BC; then, as the chain is uniform, we have

$$P : W :: BC : AC;$$

$$\therefore \frac{W}{P}=\frac{AC}{BC}, \text{ or } P=W\cdot\frac{BC}{AC}\ldots(1).$$

Now for the chain AC we may substitute a weight W resting on the plane, and for the chain BC the weight P hanging freely by a cord connected with the weight W. Thus, therefore, eq. (1) gives the relation between the weights P and W, when the weight P hanging vertically balances the tendency of the weight W to move down the plane. This is the common expression for the relation of the power and weight in the inclined plane.

Let W be moved up the plane from A to C, by the descent of P; then P will have descended a space equal to AC, whilst W will have been raised through the vertical space BC; hence we have

$$\begin{aligned}\text{Work in raising W up the plane} &= P\times AC\\ &= W\cdot\frac{BC}{AC}\times AC\\ &= W\times BC\ldots(2);\end{aligned}$$

that is to say, *the work in moving a body up an inclined plane,*

without friction, is equal to the product of the weight of the body in lbs. by the vertical height in feet through which it is raised.

If the inclination of the plane, upon which the body is moved, is small, as in the case of railway gradients, the pressure upon the plane will obviously be very nearly equal to the weight of the body; hence the work of friction in moving a body up such a plane, may be calculated after the manner explained in Art. **35.**

To find the total work, in moving the body W up the inclined plane ABC whose coefficient of friction is f.

Work in moving W from A to C

$$= \text{work due to friction} + \text{work due to gravity.}$$

$$\text{But, work due to gravity} = W \times BC;$$

and by eq. (1) Art. **35.**,

$$\text{work due to friction} = f.\, W \times AC;$$

$$\therefore \text{ Work in moving W from A to C} = fW \times AC + W \times BC \ldots (3)$$

If the body be moved down the plane, then it is evident that the work to be done will be equal to the work due to friction minus the work due to gravity; in this case, therefore, we have

$$\text{Work in moving W from C to A} = fW \times AC - W \times BC \ldots (4).$$

Problems.

1. A railway train of T tons ascends an incline, AC, which has a rise of e feet in 100 feet, with the uniform speed of m miles per hour, what must be the horse powers of the engine, the friction being p lbs. per ton?

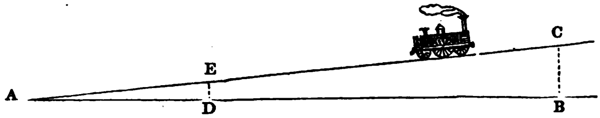


Fig. 9.

Take AC equal to the space in feet moved over by the train per min.; AE equal to 100; and draw the verticals CB and ED cutting the horizontal line AB in the points B and D; then we have

$$AC = 88m,$$

and by the similar triangles ABC and ADE, we have

$$AE : DE :: AC : BC,$$

$$\text{that is, } 100 : e :: 88m : BC,$$

$$\therefore BC = \cdot 88em.$$

Now, work per min. = work friction + work gravity;

but, work friction = resistance friction $\times$ AC

$$= p\text{T} \times 88m = 88\,mp\text{T},$$

and work gravity = wt. train in lbs. $\times$ BC

$$= 2240\,\text{T} \times \cdot 88\,em,$$

$\therefore$ Work per min. $= 88\,mp\text{T} + 2240\,\text{T} \times \cdot 88\,em$

$$= 88\,m\text{T}(p + 22\cdot 4\,e).$$

$$\therefore\ \text{N} = \frac{8}{3000}m\text{T}(p + 22\cdot 4\,e) \ldots (1).$$

If the train descends the plane, then the work due to gravity must be subtracted from the work due to friction; in this case we therefore have

$$\text{N} = \frac{8}{3000}m\text{T}(p - 22\cdot 4\,e) \ldots (2).$$

Here there are five general quantities, any four of which being given, the remaining one may be found: thus if T be required, we have

$$\text{T} = \frac{3000\,\text{N}}{8\,m(p \pm 22\cdot 4\,e)} \ldots (3),$$

where the plus or minus sign is used according as the train is ascending or descending the incline.

If m be required, we have

$$m = \frac{3000\,\text{N}}{8\,\text{T}(p \pm 22\cdot 4\,e)} \ldots (4).$$

Example 1. Let $\text{T}=80, e=2, m=15, p=8$; required N when the train ascends the incline.

Here by eq. (1), we have

$$\text{N} = \frac{8}{3000} \times 15 \times 80 \times (8 + 22\cdot 4 \times 2) = 168\cdot 96.$$

Example 2. Let $\text{N}=50$, $e=\frac{3}{4}$, $m=20$, $p=8$; required T when the train ascends the gradient.

Here by eq. (3), we have

$$\text{T} = \frac{3000 \times 50}{8 \times 20 \times (8 + 22\cdot 4 \times \frac{3}{4})} = 37\cdot 8 \text{ tons}.$$

Example 3. Let $\text{T}=100$, $e=\frac{1}{6}$, $p=8$, $\text{N}=80$; required m, when the train ascends the gradient.

Here by eq. (4), we have

$$m=\frac{3000\times 80}{8\times 100(8+22\cdot 4\times\frac{1}{6})}=25\cdot 5 \text{ miles.}$$

2. If a horse exert a traction of t lbs., what weight, w, will he pull up or down, as the case may be, a hill of small inclination which has a rise of e in 100, the coefficient of friction being f?

$$\text{Work horse over 100 feet}=100\,t,$$

$$\text{Work due to resistances over 100 ft.}=f\text{w}\times 100\pm \text{w}e,$$

$$\therefore\ f\text{w}\times 100\pm\text{w}e=100t,$$

$$\therefore\ \text{w}=\frac{100t}{100f\pm e}\ldots(1).$$

If t be required, then

$$t=f\text{w}\pm\tfrac{1}{100}\text{w}e$$
$$=\text{w}(f\pm\tfrac{1}{100}e)\ldots(2).$$

Example 1. If $t=160$, $e=3$, $f=\frac{1}{12}$; required w when the load is drawn up the hill.

Here by eq. (1), we have

$$\text{w}=\frac{100\times 160}{100\times\frac{1}{12}+3}\text{ lbs.}=12\cdot 6 \text{ cwts.}$$

Example 2. If $\text{w}=6\times 2240$, $e=2\frac{1}{2}$, $f=\frac{1}{12}$; to find t when the load is drawn up the hill.

By eq. (2), we have

$$t=6\times 2240(\tfrac{1}{12}+\tfrac{1}{100}\times 2\tfrac{1}{2})=1456 \text{ lbs.}$$

Work in Excavations and the Transport of Earth.

37. When the material is transported by means of barrows, the following method of calculating the work of excavations has been adopted by some of the most eminent French engineers.

The material is classed according to the number of pickmen necessary to keep at work a certain number of shovellers. Thus earth, of mean quality, requires 3 of the former to 2 of the latter; whereas earth, of a harder quality, will require a greater proportion of pickmen. This proportion will, of course, be fixed by the judgment of the engineer. A *relay* is the distance a barrowman will wheel a full barrow and return with an empty one, while a shoveller is filling another barrow. This distance on the horizontal line is calculated to be 120 feet. The barrow is supposed to contain one cubic foot of the material, and a shoveller to lift, on

an average, 500 cubic feet of earth per day. When the material is to be raised to the surface, this is supposed to be done by means of *ramps*, formed of planks, each 12 feet in length, and having a rise of 1 foot. The length of the relay upon the ramp is 80 feet. The *mean height* to which the material is raised, is the height to which the centre of gravity of the whole material is raised.

Problems.

1. a cubic feet of earth are to be excavated, and then conveyed to the mean distance of b feet, the material is such as to require p pickmen to s shovellers; it is required to find the number of workmen of each sort necessary to complete the work in D days, and also the total number of days' work.

$$\text{No. relays} = \frac{b}{120},$$

therefore for each shoveller we must have $\frac{b}{120}$ barrowmen.

$$\text{No. days' work shovellers} = \frac{a}{500},$$

$$\therefore \quad \text{“} \quad \text{“} \quad \text{barrowmen} = \frac{a}{500} \times \frac{b}{120},$$

$$\therefore \quad \text{“} \quad \text{“} \quad \text{pickmen} = \frac{a}{500} \times \frac{p}{s}.$$

$$\text{No. shovellers to do the work in D days} = \frac{a}{500\text{D}} \ldots (1),$$

$$\text{“ barrowmen “ “ “} = \frac{a}{500\text{D}} \times \frac{b}{120}, \ldots (2),$$

$$\text{“ pickmen “ “ “} = \frac{a}{500\text{D}} \times \frac{p}{s} \ldots (3).$$

$$\text{Total no. days' work} = \frac{a}{500} + \frac{a}{500} \cdot \frac{b}{120} + \frac{a}{500} \cdot \frac{p}{s}$$

$$= \frac{a}{500}\left(1 + \frac{b}{120} + \frac{p}{s}\right) \ldots (4).$$

Example 1. Let $a = 60{,}000$, $b = 480$, $p = 3$, $s = 2$, and $\text{D} = 20$. From eqs. (1), (2), and (3), we have

$$\text{No. shovellers} = \frac{60000}{500 \times 20} = 6,$$

$$\text{No. barrowmen}=6\times\frac{480}{120}=24,$$

$$\text{" pickmen} =6\times\frac{3}{2}=9.$$

From eq. (4), we have

$$\text{Total no. days' work}=\frac{60000}{500}\left(1+\frac{480}{120}+\frac{3}{2}\right)=780.$$

2. Required the same as in the last problem, when the earth is first to be raised to the mean height of h feet by means of ramps, and afterwards transported to the horizontal distance of b feet.

Here there will be h planks in the ramp;

$$\therefore \text{ Length of ramp}=12h;$$

$$\therefore \text{ No. relays on the ramp}=\frac{12h}{80}=\frac{3h}{20};$$

$$\therefore \text{ Total no. relays}=\frac{b}{120}+\frac{3h}{20}=\frac{b+18h}{120}.$$

$$\text{No. days' work shovellers}=\frac{a}{500},$$

$$\text{" " barrowmen}=\frac{a}{500}\times\frac{b+18h}{120},$$

$$\text{" " pickmen}=\frac{a}{500}\times\frac{p}{s}.$$

$$\therefore \text{ Total no. days' work}=\frac{a}{500}\left(1+\frac{b+18h}{120}+\frac{p}{s}\right).$$

3. A cubic feet of earth are to be excavated and then transported, by means of an engine, to the mean distance of m miles. The engine travels at the rate of r miles per hour with the full waggons, and returns with the empty ones at the rate of r_1 miles. There are a cubic feet of earth conveyed each journey; and the material is such as to require p pickmen to s shovellers. It is required to find the time in which the excavation will be completed, the number of men of each sort necessary to keep the engine going, and the cost of the excavation (exclusive of the engine), allowing s shillings per day of 10 hours long to each workman; and that a shoveller can lift 500 c. ft. of the earth per day?

$$\text{Time each journey}=\frac{m}{r}+\frac{m}{r_1}=\frac{m(r+r_1)}{rr_1};$$

$$\therefore \text{ No. journeys per day} = 10 \div \text{time each journey}$$

$$= \frac{10rr_1}{m(r+r_1)};$$

$$\therefore \text{ No. c. ft. conveyed per day, } n = \frac{10arr_1}{m(r+r_1)};$$

$$\therefore \text{ No. days to complete the transport, \&c.} = \frac{\text{A}}{n} \ldots (1).$$

$$\text{No. shovellers} = \frac{n}{500} \ldots (2).$$

$$\therefore \text{ No. pickmen} = \frac{n}{500} \times \frac{p}{s} \ldots (3).$$

$$\text{Total no. days' work} = \frac{\text{A}}{500} + \frac{\text{A}}{500} \times \frac{p}{s},$$

$$\therefore \text{ Cost in £} = \frac{\text{A}.\text{S}}{10000}\left(1 + \frac{p}{s}\right) \ldots (4).$$

Example. Required the number of men of each sort, when $m=4$, $r=8$, $r_1=12$, $a=648$, $p=2$, $s=3$.

Here by eqs. (2) and (3), we have

$$\text{No. shovellers} = \frac{n}{500} = \frac{10 \times 648 \times 8 \times 12}{500 \times 4(8+12)} = 15{\cdot}552.$$

$$\text{No. pickmen} = 15{\cdot}552 \times \frac{2}{3} = 10{\cdot}368.$$

Work in overcoming the Resistance of Fluids.

38. The resistance of the atmosphere to the motion of a body, within certain limits, varies as the square of the velocity: thus, if R lbs. be the resistance of the atmosphere to a body when moving at the rate of m miles per hour, then $\text{R} \times \left(\frac{m_1}{m}\right)^2$ will express the resistance when the body moves at the rate of m_1 miles per hour.

In a railway train the resistance of the air also increases with the frontage of the carriages as well as with the length of the train. If the frontage be constant, the resistance may be assumed to vary with the number of the carriages: thus if we take R (as determined by experiment) to represent the lbs. resistance of the atmosphere to a given train of n carriages moving at the rate of 10 miles per hour; then the resistance to n_1 carriages moving at the same rate may be expressed by

$$\text{R}\{1 + (n_1 - n)k\},$$

where k is a constant which must be determined by experiment.

Let m be the rate of the n_1 carriages per hour, then

$$\text{Resistance air} = \text{R}\{1+(n_1-n)k\}\left(\frac{m}{10}\right)^2,$$

$\therefore$ Work due to the resistance of the air per min.

$$=\frac{5280m}{60}\times\text{R}\{1+(n_1-n)k\}\left(\frac{m}{10}\right)^2$$

$$=\cdot 88\,m^3\text{R}\{1+(n_1-n)k\},$$

where it will be observed that the work varies as the cube of the speed.

And the horse powers due to the resistance of the air

$$=\frac{8}{300000}m^3\text{R}\{1+(n_1-n)k\}.$$

Adding this to the horse powers due to gravity and friction, given in eqs. (1) or (2), Problem 1, Art. 36., we get

$$\text{N}=\frac{8}{3000}\left[m\text{T}(p\pm 22\cdot 4e)+\frac{m^3\text{R}}{100}\{1+(n_1-n)e\}\right]\ \ldots\ (1).$$

If $n_1=n$, or what amounts to the same thing, if the resistance depending upon the length of the train be neglected, then this equality becomes

$$\text{N}=\frac{8}{3000}\left\{m\text{T}(p\pm 22\cdot 4e)+\frac{m^3\text{R}}{100}\right\}\ \ldots\ (2).$$

Taking $e=0$, and multiplying by 33000, this equality becomes

$$\text{U}=88\left\{mp\text{T}+\frac{m^3\text{R}}{100}\right\}\ \ldots\ (3),$$

which is the work per min. on the level rail, due to the resistance of friction and the air.

If T be required, from eq. (2), we get

$$\text{T}=\frac{3000\text{N}-\cdot 08\,m^3\text{R}}{8\,m(p\pm 22\cdot 4e)}\ \ldots\ (4).$$

If the rail be level, then e in these expressions becomes 0.

Example 1. Let $\text{T}=100$, $m=40$, $p=8$, $e=0$, $\text{R}=33$; required N.

Here by eq. (2), we have

$$\text{N}=\frac{8}{3000}\left\{40\times 100\times 8+\frac{40^3\times 33}{100}\right\}=141.$$

Example 2. Let $\text{N}=120$, $m=30$, $p=8$, $e=0$, $\text{R}=33$; required T.

Here by eq. (4), we get

$$T=\frac{3000\times 120-\cdot 08\times 30^3\times 33}{8\times 30\times 8}=150\cdot 3 \text{ tons.}$$

Work of a Fall of Water.

39. When water, or any body falls from a given height, the work which it is capable of performing is obviously equal to that which would be done upon it in raising it to the height from which it has fallen.

When a fall of water is employed to drive a water-wheel, or any other hydraulic machine, whose modulus is given, the work done upon the machine is equal to the weight of the water in lbs. × its fall in feet × the modulus of the machine.*

Problems.

1. The breadth of a stream is b feet, depth a feet, mean velocity of the water v feet per minute, and the height of the fall h feet; required the horse powers, N, of the water-wheel whose modulus is M.

Wt. water going over the fall per min. $=62\cdot 5\,abv$,

$$\therefore \text{ Work water per min.}=62\cdot 5abvh,$$

$$\therefore \text{ Work wheel per min.}=62\cdot 5\,abvh\text{M} \ldots (1);$$

$$\therefore \text{N}=\frac{62\cdot 5\,abvh\text{M}}{33000}\ldots (2).$$

If h be required, then

$$h=\frac{33000\,\text{N}}{62\cdot 5\,abv\text{M}}\ldots (3).$$

Example 1. Let $a=3$, $b=4$, $v=15$, $h=20$, and $\text{M}=\cdot 7$; required N.

Here by eq. (2), we have

$$\text{N}=\frac{62\cdot 5\times 3\times 4\times 15\times 20\times\cdot 7}{33000}=4\cdot 77.$$

Example 2. Let $\text{N}=3$, $a=4$, $b=2$, $v=20$, $\text{M}=\cdot 68$; required h.

Here by eq. (3) we have

$$h=\frac{33000\times 3}{62\cdot 5\times 4\times 2\times 20\times\cdot 68}=14\cdot 5 \text{ ft.}$$

* Here the work accumulated in the water at its delivery on the wheel is neglected as being comparatively small.

2. To determine the number of cubic feet of water, A, which the wheel in the last problem will pump per min., from the bottom of the fall to the height of h_1 feet.

$$\text{Work in pumping water per min.} = 62.5\,\text{A}h_1.$$

But this must be equal to the work of the wheel per min.; hence we have from eq. (1) of the last problem,

$$62{\cdot}5\,\text{A}h_1 = 62{\cdot}5\,abvh\text{M},$$

$$\therefore\ \text{A} = \frac{abvh\text{M}}{h_1} \ldots (1).$$

3. To determine the number of cubic feet of water, A_1, which the wheel will pump per min. from the top of the fall to the same height as in the last problem.

No. c. ft. water going over the fall per min. $= abv - \text{A}_1$,

$$\therefore\ \text{Work wheel per min.} = 62{\cdot}5\,(abv - \text{A}_1)h\text{M}.$$

$$\text{Work pumping water per min.} = 62{\cdot}5\,\text{A}_1(h_1 - h).$$

$$\therefore\ 62{\cdot}5\,\text{A}_1(h_1 - h) = 62{\cdot}5(abv - \text{A}_1)h\text{M},$$

$$\therefore\ \text{A}_1 = \frac{abvh\text{M}}{h_1 - h(1-\text{M})} \ldots (1)$$

This expression is greater than that of eq. (1) Problem 2; for M is always less than unity. Hence it appears that it would be more advantageous to pump water from the *top* of the fall.

WORK OF STEAM.

WORK OF STEAM, HAVING A MEAN PRESSURE.

40. If steam in the cylinder AD exert any constant, or mean effective pressure upon the piston AB, say of L lbs. per square inch, then if a weight of L lbs. be placed upon every inch of surface in the piston, the elastic vapour would just be able to move the piston with its weights through the length of the stroke in opposition to gravity; therefore the work performed upon 1 inch of the piston in one stroke will be the pressure of the steam upon 1 inch multiplied by the number of feet in the stroke, and the work upon the whole piston will be the work upon 1 inch multiplied by the number of inches in the whole piston. Moreover the work done upon the whole piston in one stroke multiplied by the number of strokes performed per minute will give

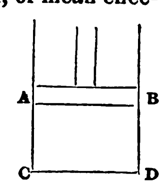


Fig. 10.

the work done per minute: Thus let K = the area of the piston in sq. inches, l = the length of the stroke in feet, and U = the work done per min., and N = the effective horse powers; then

Effective work steam on 1 inch piston in one stroke $= L \times l$,

" " on K inches " " $= L \times l \times K$,

Effective work per min. or $U = L \times l \times K \times n \ldots (1)$,

$$\text{and } N = \frac{L \times l \times K \times n}{33{,}000} \ldots (2).$$

If we put E = the volume of steam, in cubic feet, discharged per minute, then

$$E = \frac{l \times K \times n}{144},$$

and by substitution eqs. (1) and (2) becomes

$$U = 144 L E \ldots (\alpha).$$

$$N = \frac{144 L E}{33{,}000} \ldots (\beta).$$

This mean effective pressure of the steam, L, is also called the useful load of the engine. The whole pressure of the steam, P, must not only overcome the pressure arising from the useful load, but also certain useless resistances, which may be treated after the following manner.

In the high pressure engine, the pressure of the atmosphere,—about 15 lbs. per sq. inch,—is opposed to the pressure of the steam. Besides this, a considerable portion of the pressure of the steam is required to overcome the friction of the parts of the engine. As a mean estimate, 1 lb. to the sq. inch is allowed for the friction due to the engine when unloaded; and an additional friction of $\frac{1}{7}$ of the effective pressure, or useful load, for the resistance necessary to overcome the friction of the loaded engine.

Thus, in this case, we have

$$P = L + \tfrac{1}{7} L + 1 + 15.$$

In the condensing engine, the pressure of the vapour in the condenser (estimated at a maximum about 4 lbs. per sq. in. of the piston) must be used in the place of the atmospheric pressure Thus, in this case, we have

$$P = L + \tfrac{1}{7} L + 1 + 4;$$

Or generally if we put f_1 for the constant ratio $\frac{1}{7}$, and f_2 for

$1+15$ or $1+4$, as the case may be, the resistances of the unloaded piston; then

$$P = L + f_1 L + f_2 \ldots (3);$$

$$\therefore\ L = \frac{P - f_2}{1 + f_1} \ldots (4).$$

Problems.

1. To determine the effective horse powers of an engine, N, in in terms of the mean pressure of the steam, P, and the coefficients of friction, &c.

Substituting in eq. (2), the value of L as expressed in eq. (4), we find

$$N = \frac{(P - f_2)\, l n K}{33000(1 + f_1)} \ldots (5).$$

If P be required, we have from this equality

$$P = \frac{33000 N(1 + f_1)}{l n K} + f_2 \ldots (6).$$

Example 1. Suppose the engine to be high pressure, and let $K=500$, $l=6$, $P=40$, $n=16$, $f_1=\frac{1}{7}$, $f_2=1+15$; required N.

By eq. (5), we have

$$N = \frac{(40-16)\times 6\times 16\times 500}{33000(1+\frac{1}{7})} = 30 \cdot 5.$$

Example 2. Let $K=3000$, $l=10$, $n=16$, $N=120$, the constants being the same as in the foregoing example; to find P.

By eq. (6), we have

$$P = \frac{33000\times 120(1+\frac{1}{7})}{10\times 16\times 3000} + 16 = 25 \cdot 4 \text{ lbs.}$$

2. What must be the pressure, P, of the steam, so that the engine may pump A cubic feet of water per min. from a mine whose depth is a fathoms, the modulus pump being M?

By problem 5, Art. **34.**,

$$N = \frac{375\, a A}{33000 M}$$

Substituting this in eq. (6) of the foregoing problem, we get

$$P = \frac{375\, a A(1 + f_1)}{l n K M} + f_2 \ldots (1).$$

Example. Let $K=1000$, $l=8$, $n=20$, $a=120$, $A=80$, $M=\cdot 66$,

and the constants f_1 and f_2 as in the preceding examples; to find P.

$$P=\frac{375\times 120\times 80(1+\frac{1}{7})}{8\times 20\times 1000\times \cdot 66}+16=54\cdot 9 \text{ lbs.}$$

Work of Steam, with a Mean Pressure, considered in relation to the Water evaporated.

41. The true source of work in the steam-engine is the evaporating power of the boiler. The magnitude of the work not only depends upon the quantity of water evaporated in a given time, but also upon the temperature, and consequently the pressure at which the steam is formed. Experimental tables have been formed, giving the relation of the volume and pressure of steam raised from a cubic foot of water; these tables will enable us to find the volume of the steam when its pressure and the volume of the water are given, and *vice versâ.*

The following table is given by Pambour:—

Table.

Volume of a cubic foot of water in the form of steam at the corresponding pressures and temperatures.

P. Total pressure, in pounds, per square inch.	Corresponding temperature by Fahrenheit's thermometer.	V. Volume of the steam compared to the volume of the water that has produced it.	P. Total pressure, in pounds, per square inch.	Corresponding temperature by Fahrenheit's thermometer.	V. Volume of the steam compared to the volume of the water that has produced it.
1	102·9	20954	19	225·6	1342
2	126·1	10907	20	228·3	1280
3	141·0	7455	21	231·0	1224
4	152·3	5695	22	233·6	1172
5	161·4	4624	23	236·1	1125
6	169·2	3901	24	238·4	1082
7	176·0	3380	25	240·7	1042
8	182·0	2985	26	243·0	1005
9	187·4	2676	27	245·1	971
10	192·4	2427	28	247·2	939
11	197·0	2222	29	249·2	909
12	201·3	2050	30	251·2	882
13	205·3	1903	31	253·1	855
14	209·0	1777	32	255·0	831
15	213·0	1669	33	256·8	808
16	216·4	1572	34	258·6	786
17	219·6	1487	35	260·3	765
18	222·6	1410	36	262·0	746

P. Total pressure, in pounds, per square inch.	Corresponding temperature by Fahrenheit's thermometer.	V. Volume of the steam compared to the volume of the water that has produced it.	P. Total pressure, in pounds, per square inch.	Corresponding temperature by Fahrenheit's thermometer.	V. Volume of the steam compared to the volume of the water that has produced it.
37	263·7	727	74	308·0	386
38	265·3	710	75	308·9	381
39	266·9	693	76	309·9	377
40	268·4	677	77	310·8	372
41	269·9	662	78	311·7	368
42	271·4	647	79	312·6	364
43	272·9	634	80	313·5	359
44	274·3	620	81	314·3	355
45	275·7	608	82	315·2	351
46	277·1	596	83	316·1	348
47	278·4	584	84	316·9	344
48	279·7	573	85	317·8	340
49	281·0	562	86	318·6	337
50	282·3	552	87	319·4	333
51	283·6	542	88	320·3	330
52	284·8	532	89	321·1	326
53	286·0	523	90	321·9	323
54	287·2	514	91	322·7	320
55	288·4	506	92	323·5	317
56	289·6	498	93	324·3	313
57	290·7	490	94	325·0	310
58	291·9	482	95	325·8	307
59	293·0	474	96	326·6	305
60	294·1	467	97	327·3	302
61	294·9	460	98	328·1	299
62	295·9	453	99	328·8	296
63	297·0	447	100	329 6	293
64	298·1	440	105	333·2	281
65	299·1	434	120	343·3	249
66	300·1	428	135	352·4	224
67	301.2	422	150	360·8	203
68	302·2	417	165	368·5	187
69	303·2	411	180	375·6	173
70	304·2	406	195	382·3	161
71	305·1	401	210	388.6	150
72	306·1	396	225	394·6	141
73	307·1	391	240	400·2	133

The Author has given a general formula expressing this relation with remarkable exactness, between the range of 5 and 200 lbs. pressure. This formula is

$$\text{V}=a+b\,\text{P}^{a}\ldots\ (1);$$

where V is the volume of a cubic foot of water in the form of steam at P lbs. pressure, $a=12\cdot5$, $b=20570$, and $\alpha=-\cdot9301$.

In order to show the application of this formula, let it be required to find the volume of a cubic foot of water in the form of steam at 40 lbs. pressure per square inch.

Substituting the value of the constants, we have

$$V=12\cdot5+20570P^{-\cdot9301} \ldots (2).$$

In the case proposed $P=40$,

$$\therefore\ V=12\cdot5+20570\times40^{-\cdot9301}.$$

To calculate $20570\times40^{-\cdot9301}$, we have by logarithms,

$$\begin{aligned}\log.(20570\times40^{-\cdot9301})&=\log.20570-\cdot9301\times\log.40\\&=4\cdot31323-\cdot9301\times1\cdot60206\\&=2\cdot823154;\end{aligned}$$

but this number is the logarithm of 665·5, which is the value of the quantity required. Substituting this, we have

$$V=12\cdot5+665.5=678.$$

Now the volume in the table corresponding to 40 lbs. pressure is 677.

Problems.

42. Let K=the area of the piston in sq. inches; l=the length of the stroke of the piston, or more strictly speaking the length of cylinder occupied by the steam; P=the pressure of the steam per sq. in. of the piston; c=the cubic feet of water evaporated per min.; V=the volume of a cubic foot of water in the form of steam at P lbs. pressure; and so on as before.

1. To find n the number of strokes per min. given K, l, P, and c.

When P is given V may be found from the table, or it may be calculated by the formula.

$$\text{Vol. steam evaporated per min.}=cV,$$

$$\text{vol. steam used each stroke}=\frac{Kl}{144},$$

$$\therefore\ \text{,, ,, ,, in } n \text{ strokes}=\frac{nKl}{144},$$

but this is also the steam used per min.,

$$\frac{nKl}{144}=cV,$$

$$\therefore\ n=\frac{144c\mathrm{V}}{\mathrm{K}l}\ldots(1).$$

2. To find c, when n &c. are given.

$$c=\frac{\mathrm{K}ln}{144\mathrm{V}}\ldots(2).$$

3. To find V and P, when K, l, n, and c are given.

$$\mathrm{V}=\frac{\mathrm{K}ln}{144c}\ldots(3).$$

Whence P may be found from the table, or by substitution in eq. (1), Art. **41.**

Example. If $\mathrm{K}=120$, $l=6$, $n=24$, and $c=\cdot 2$, required P.

$$\mathrm{V}=\frac{24\times 120\times 6}{144\times\cdot 2}=600,$$

whence we find from the table that P=46 lbs. nearly.

4. To find the useful load L the pressure P being given.

This is given in eq. (4), Art. **40.**

5. To find the useful horse powers, N, when K, l, n, and c are given, the clearance being neglected.*

By eq. (2), Art. **40.**,

$$\mathrm{N}=\frac{\mathrm{LK}ln}{33000}\ldots(4),$$

substituting the value of L, given in eq. (4), Art. **40.**, we get

$$\mathrm{N}=\frac{\mathrm{P}-f_2}{1+f}\cdot\frac{\mathrm{K}ln}{33000}\ldots(5).$$

Now when K, l, n, and c, are given, P may be found by Prob. 3.

To find N, when P, and c are given.

Substituting in eq. (4) of the last problem, the value of n given in eq. (1), Prob. 1., and reducing, we get

$$\mathrm{N}=\frac{144c\mathrm{LV}}{33000}\ldots(6);$$

eliminating L by eq. (4), Art. **40.**, we get

$$\mathrm{N}=\frac{144c\mathrm{V}(\mathrm{P}-f_2)}{33000(1+f_1)}\ldots(7).$$

Where V may be determined from P by the table; or substituting the value of V given in the formula, we get

* The clearance is that space in the cylinder lying beneath the piston, at the lowest point of its stroke.

$$N = \frac{144c(a+bP^a)(P-f_2)}{33000(1+f_1)} \ldots (8).$$

6. To find the useful work U performed per minute.

In this case it is only requisite to multiply any of the foregoing expressions for N by 33000.

It will be observed that the work, as expressed in these formulæ, is entirely independent of the form or volume of the cylinder; in fact the work depends upon P and c only, the other quantities being constants for any given engine.

When the pressure P remains constant, the work increases with c the quantity of water evaporated; and when c remains constant, it may be shown from eq. (8) that the work increases with P the pressure at which that water is converted into steam. Now it has been found, by experiment, that whatever may be the pressure at which the steam is formed, the quantity of fuel necessary to evaporate a given volume of water is always the same. Hence it follows that it is most advantageous to employ steam of a high pressure.*

Example 1. In a high pressure engine, let $P=50$, $c=\cdot335$, and the coefficients of friction as before; required L and N.

By eq. (4), Art. **40.**,

$$L = \frac{50-16}{1+\frac{1}{7}} = 29\cdot75 \text{ lbs.}$$

By the table $V=552$ when $P=50$, therefore by eq. (6), we have

$$N = \frac{144 \times \cdot335 \times 29\cdot75 \times 552}{33000} = 24.$$

Example 2. In a high pressure engine, given $K=144$, $l=3$, $P=48$, $n=20$, &c.; required L, c, and N.

By eq. (4), Art. **40.**,

$$L = \frac{48-16}{1+\frac{1}{7}} = 28 \text{ lbs.}$$

By eq. (4),

$$N = \frac{28 \times 144 \times 3 \times 20}{33000} = 7\cdot3.$$

By the table $V=573$, when $P=48$, therefore by eq. (2), Prob. 2., we have

$$c = \frac{144 \times 3 \times 20}{144 \times 573} = \cdot104.$$

* This principle was first demonstrated by the Author in the "Mechanics' Magazine" for the year 1841.

7. To determine the duty of an engine.

The DUTY of an engine is the number of units of work which it is capable of performing with a bushel of coals.

Let U=the useful work performed by the engine per min.; D=the duty of the engine; c=the number of cubic feet of water evaporated per min.; c_1=the number of cubic feet of water which 1 bushel of coals has been found by experiment to evaporate.

Work done by c cubic ft. water=U;

∴ " " c_1 " " $=\frac{c_1 \text{U}}{c}$;

but this work is done by 1 bushel of coals,

$$\therefore \text{D}=\frac{c_1 \text{U}}{c} \ldots (9).$$

This expresses the duty of an engine in terms of the useful work, the water evaporated, and the experimental constant c_1.

Again, let b=the number of bushels of coals actually consumed by the engine per hour; then, in this case, we have

Work done by b bushels=60 U;

∴ " " 1 bushel $=\frac{60\text{U}}{b}$;

$$\therefore \text{D}=\frac{60\text{U}}{b} \ldots (10).$$

For the value of U in these expressions, see Prob. 6.

Example. Required the duty of the engine of Example 2., page 47., supposing 1 bushel of coals to evaporate 11·5 cubic feet of water.

Now in eq. (9), we have c_1=11·5, and we have found, in the example referred to,

$$\text{U}=28\times144\times3\times20,\ c=\cdot104;$$

$$\therefore \text{D}=\frac{11\cdot5\times28\times144\times3\times20}{\cdot104}=27 \text{ millions nearly.}$$

Locomotive Engine.

43. In a locomotive engine, the pressure of the steam has to overcome the resistance due to the blast pipe, in addition to the resistances given in Art. **40.** Now it has been found, by experiment, that the resistance due to the blast pipe increases with the speed of the engine: thus if m=the speed of the engine in miles

per hour, p_1=the resistance due to the blast pipe when the speed is 10 miles per hour, then

$$\text{Resistance due to the blast pipe} = \tfrac{1}{10} m p_1.$$

Adding this to the right hand member of eq. (3), Art. **40.**, we get

$$\text{P} = \text{L} + f_1 \text{L} + f_2 + \tfrac{1}{10} m p_1 \ldots (1);$$

here L is the effective pressure of the steam per square inch of the piston tending to turn the crank fixed to the driving wheel of the engine.

The notation of Art. **38.** is adopted in the following problems.

Problems.

1. To find the pressure of the steam P, when m, T, p, p_1, and R are given, and also the diameter d of the driving wheel.

Resistance to the train=resistance friction+resistance air

$$= p\text{T} + \frac{\text{R}m^2}{100}.$$

Space moved over by the train in 1 revo. driving wheel$=\pi d$;

$$\therefore \text{ Work in 1 revo. driving wheel} = \left(p\text{T} + \frac{\text{R}m^2}{100}\right)\pi d.$$

Because each piston makes two strokes whilst the driving wheel performs 1 revolution, we have

$$\text{Eff. work steam in 1 revo. driving wheel} = \text{L} \times \text{K} \times l \times 4.$$

Now as the work applied is equal to the work done, we have

$$\text{L} \times \text{K} \times l \times 4 = \left(p\text{T} + \frac{\text{R}m^2}{100}\right)\pi d;$$

eliminating L between this equation and eq. (1), we get

$$\text{P} = \frac{(1+f_1)\left(p\text{T} + \frac{\text{R}m^2}{100}\right)\pi d}{4 l\text{K}} + f_2 + \tfrac{1}{10} m p_1 \ldots (2).$$

In this expression, we have five general quantities which may be supposed to change, viz. P, T, m, d, and lK the volume of the cylinder; and any four of which being given the remaining one may be found; for example, let T be required, then

$$\text{T} = \frac{4 l\text{K}(\text{P} - f_2 - \tfrac{1}{10} m p_1)}{(1+f_1)\pi d p} - \frac{\text{R}m^2}{100 p} \ldots (3).$$

2. To find the number of cubic feet c of water evaporated, when P, m, l, K, and d are given.

$$\text{No. revo. driving wheel per min.}=\frac{5280\,m}{60}\div\pi d$$

$$=\frac{88\,m}{\pi d};$$

$$\therefore\ \text{No. strokes piston per min.}=\frac{88\,m}{\pi d}\times 4;$$

$$\therefore\ \text{Vol. steam used per min.}=\frac{Kl}{144}\times\frac{88\,m}{\pi d}\times 4=\frac{22\,mlK}{9\pi d}.$$

$$c=\frac{\text{vol. steam used per min.}}{\text{vol. 1 c. ft. water in the form of steam}}$$

$$=\frac{22\,mlK}{9\pi dV}\ \ldots\ (4);$$

where V will be found from the table when P is given. Or substituting the value of V given in eq. (1), Art. 41.,

$$c=\frac{22\,mlK}{9\pi d(a+bP^{n})}\ \ldots\ (5).$$

The value of c substituted in eq. (10), Art. 42., will give the duty of the engine.

3. To find the speed of the train, m, when P, l, K, d, and c are given.

Here from eq. (4), we get

$$m=\frac{9\pi cdV}{22\,lK}\ \ldots\ (6).$$

Example 1. Required P and c, when $T=100$, $m=20$, $p=7$, $R=33$, $d=5$, $K=110$, $l=\frac{4}{3}$, $p_1=1{\cdot}75$, and the coefficients f_1 and f_2 as before.

Here by eq. (2), we have

$$P=\frac{(1+\frac{1}{7})\left(7\times 100+\frac{33\times 20^2}{100}\right)3{\cdot}1415\times 5}{4\times\frac{4}{3}\times 110}+16+\tfrac{1}{10}\times 20\times 1{\cdot}75$$

$$=25{\cdot}45+16+3{\cdot}5=45\text{ lbs. nearly.}$$

Now from the table we find $V=608$ when $P=45$; hence by eq. (4), we have

$$c=\frac{22\times 20\times\frac{4}{3}\times 110}{9\times 3{\cdot}1416\times 5\times 608}={\cdot}74\text{ cubic ft. water.}$$

Example 2. Required m, when $P=50$, $l=1\frac{1}{3}$, $K=90$, $d=5$, and $c=\cdot 7$.

Now by the table $V=552$, when $P=50$.

By eq. (6), we have

$$m=\frac{9\times 3\cdot 1416\times\cdot 7\times 5\times 552}{22\times 1\frac{1}{3}\times 90}=20\cdot 7 \text{ miles.}$$

Example 3. Required T and c, when $K=80$, $l=1\frac{1}{4}$, $P=48$ $d=5$, $m=30$, and the resistances being the same as in Example 1.

By eq. (3), we have

$$T=\frac{4\times 1\frac{1}{4}\times 80\times(48-16-\frac{1}{10}\times 30\times 1\cdot 75)}{(1+\frac{1}{7})\times 3\cdot 1416\times 5\times 7}-\frac{33\times 30^2}{100\times 7}$$

$$=42\cdot 6 \text{ tons.}$$

Now by the table $V=573$ when $P=48$.

By eq. (4), we have

$$c=\frac{22\times 30\times 1\frac{1}{4}\times 80}{9\times 3\cdot 1416\times 5\times 573}=\cdot 81.$$

4. To find the duty of the engine.

The useful work U done per min. is given in eq. (3), Art. 38., and the water evaporated per min., c, is given in eq. (4), Problem 2.; substituting the values of these quantities in eq. (9) Art. 42., and reducing, we get

$$D=\frac{c_1\cdot U}{c}$$

$$=\frac{36\pi c_1 d V\left(p T+\frac{m^2 R}{100}\right)}{l K}\ .\ .\ .\ (7).$$

Work developed by the Condensation of Steam.

44. When water is raised into steam at the boiling temperature, or 212°, its volume is increased 1710 times, or a cubic inch of water will very nearly form a cubic foot of steam. Now if steam at this temperature be allowed to enter the lower part of the cylinder, then the pressure beneath the piston will just counterpoise the pressure of the air upon the piston, and a small additional force will cause the piston to rise. If, then, the steam be condensed by a jet of cold water, a vacuum will be formed, and the piston will be pressed downwards with the whole weight of

the atmosphere resting upon the surface of the piston. But it has been found that a perfect vacuum cannot be formed in this way, because water gives off vapour at all temperatures. Thus, at the temperature of 150°, the pressure of the vapour is 4 lbs. And if 14·7 lbs. be taken as the mean pressure of the atmosphere, upon 1 inch of surface, we shall have, by the condensation of steam, upon an average, an effective pressure of 14·7 lbs. − 4 lbs. = 10·7 lbs. upon each inch of the piston.

Problems.

1. To determine the work developed by the condensation of a cubic foot of water in the state of steam at the boiling temperature, and also the duty of an atmospheric engine using steam in this manner.

Let P_1 = the pressure of the atmosphere, P_2 = the elasticity of the vapour after condensation, $\frac{K\,l}{144} = 1710$ the volume of the cylinder; then

$$\text{Work of cond}^{n}\text{. of 1 c. ft. steam} = (P_1 - P_2)\,K\,l$$
$$= (P_1 - P_2) \times 144 \times 1710 \ldots (1).$$
$$D = c_1\,(P_1 - P_2) \times 144 \times 1710 \ldots (2).$$

Example. If we take $P_1 = 14{\cdot}7$, $P_2 = 4$, and $c_1 = 11{\cdot}5$; then we have

$$\text{Work 1 c. ft. water} = (14{\cdot}7 - 4) \times 144 \times 1710.$$
$$= 2634768.$$

$$\text{And, } D = 11{\cdot}5 \times 2634768 = 30 \text{ millions nearly.}$$

2. Given the number of cubic feet of water, c, evaporated per min., to find the horse powers N.

$$U = c\,(P - P^2) \times 144 \times 1710,$$
$$\therefore\ N = \frac{c\,(P_1 - P_2) \times 144 \times 1710}{33000} \ldots (3).$$

Example. Let $c = {\cdot}34$, then we find $N = 30$.

WORK OF STEAM USED EXPANSIVELY.

45. When steam is used expansively, it is allowed to enter the cylinder for only a part of the stroke, and then, for the remaining portion, the piston is moved by the expansive force of the steam.

This is the most economical way of employing steam power; for all, or nearly all, the available work is taken out of the elastic vapour before it is condensed. Now when the volume of steam,—or any elastic fluid,— is increased, its elasticity or pressure is decreased nearly in the same ratio; that is, if its volume is increased two times, its pressure will be about one-half of what it was at first, and so on. This is called Marriotte's law. Let the steam be cut off when the piston is at CD, and let the remaining part of the stroke be divided into any even number of parts; then the pressure of the steam upon the piston when it arrives at the different lines, forming the division, may be ascertained by the law just explained.

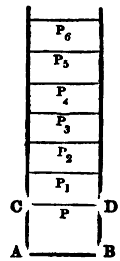


Fig. 11.

And the work done by this variable pressure may be found by formula (1) or (2), Art. **33.**; where P is the pressure of the steam when admitted into the cylinder; P_1, P_2, . . . P_n, the pressure at the end of the first, second, &c. divisions; S the space through which the steam acts expansively. The work done before the steam is cut off, will be performed with a constant pressure P. The work in the former case, is said to be done by the steam acting expansively, and in the latter case it is done by the steam acting uniformly. The work done exansively added to the work done uniformly, will obviously be equal to the whole work of the steam in each stroke.

Let u^1=the whole work of the steam done on each inch of the piston in one stroke; h=the whole stroke of the piston; l=the length of cylinder occupied by the steam at the moment the communication with the boiler is cut of; e=the clearance, or the space between the piston at its lowest point of the stroke and the bottom of the cylinder, so that $l-e$ is the space through which the pressure of the steam acts uniformly; a=the common distance between the divisions into which S is divided, so that $a=\frac{S}{n}$; then

$$u^1 = \text{work expansively} + \text{work uniformly}$$
$$= u + P(l-e) \quad . \; . \; . \quad (1),$$

where u is given in eq. (1) or (2), Art. **33.**

To find the values of the pressures at the different points of the stroke, we have by Marriotte's law,

$$\text{New pressure} = \frac{\text{original vol.}}{\text{new vol.}} \times \text{original pressure.}$$

Now when the pressure of the steam is P, its volume is l, when P_1 its volume is $l+a$, when P_2 its volume is $l+2a$, and so on; hence we have

$$P_1=\frac{l}{l+a}P,\ P_2=\frac{l}{l+2a}P,\ P_3=\frac{l}{l+3a}P,\ \ldots,\ P_n=\frac{l}{l+na}P=\frac{l}{h+e}P.$$

Substituting these values in eqs. (1) and (2), Art. 33., and reducing, we get

$$u=alP\left\{\tfrac{1}{2}\left(1+\frac{1}{h+e}\right)+\frac{1}{l+a}+\frac{1}{l+2a}+\ldots+\frac{1}{l+\overline{n-1}\,a}\right\}\ldots(2);$$

$$\text{or, } u=\frac{alP}{3}\left\{1+\frac{1}{h+e}+4\left(\frac{1}{l+a}+\frac{1}{l+3a}+\ldots+\frac{1}{l+\overline{n-1}\,a}\right)\right.$$

$$\left.+2\left(\frac{1}{l+2a}+\frac{1}{l+4a}+\ldots+\frac{1}{l+\overline{n-2}\,a}\right)\right\}\ldots(3).$$

Either of these values of the work done expansively substituted in eq. (1) will give the work upon one inch of the piston in one stroke. It will be observed that eq. (3) is more accurate than eq. (2).

If P' be put for the mean pressure of the steam, we have

$$P'\times h=u',\ \therefore\ P'=\frac{u'}{h}\ldots(4).$$

Substituting this value for the mean pressure of the steam for P in eq. (4), Art. 40., we get

$$L=\frac{\frac{u'}{h}-f_2}{1+f_1}=\frac{u'-hf_2}{h(1+f_1)}\ldots(5),$$

which is the useful load, where it is to be observed that f_2 applies to the case of the condensing engine.

Taking l according to the signification here affixed to it, eqs. (1), (2), and (3) of Art. 42. express the values of n, c, and V, as applied to the condensing engine.

For the useful horse powers, we have

$$N=\frac{LKhn}{33000}\ldots(6).$$

Substituting the value of n given in eq. (1), Art. 42., and reducing, we get

$$N=\frac{144chLV}{33000l}\ldots(7),$$

where L is given in eq. (5), and V is found from the table when P, the pressure of the steam as it is admitted into the cylinder, is given.

To calculate U, *the work done expansively, by the experimental table.*

46. As the law of Marriotte does not exactly give the relation of the volume and pressure of steam, it may be requisite, when great accuracy is to be attained, that the relation should be taken from the experimental table given at page 43. In order to adapt eqs. (1) and (2) to this mode of calculation, let E=the volume of the steam admitted into the cylinder at P pressure; C=the number of cubic ft. of water from which this steam is formed; l=the length of cylinder which this steam occupies; V=the volume of a cubic foot of water in the form of steam at P pressure; E′, l', and V′ the corresponding notation when the steam is at P′ pressure. Now it is an experimental fact, that the law connecting V and P holds true for the steam in the cylinder as well as for the steam in the boiler where it is in contact with the water from which it is generated; hence we have

$$V \times C = E, \text{ and } V' \times C = E';$$

$$\therefore \frac{V'}{V} = \frac{E'}{E} = \frac{l'}{l};$$

$$\therefore V' = \frac{l'V}{l} \ldots (1).$$

This gives the value of V′ in terms of P, for V is found from the table when P is given; and in like manner P′ is found from the table from the value of V′.

From eq. (1) we have

$$V_1 = \frac{(l+a)V}{l} = V + \frac{aV}{l},$$

$$V_2 = \frac{(l+2a)V}{l} = V + 2 \cdot \frac{aV}{l},$$

$$V_3 = \frac{(l+3a)V}{l} = V + 3 \cdot \frac{aV}{l},$$

&c.=&c.

Now P is the pressure at V volume, P_1 at V_1, P_2 at V_2, and so on; therefore from the table P_1, P_2, &c., may be found from calculated values of V_1, V_2, &c. These values of P_1, P_2, &c., substituted in eq. (3), Art. **45.**, will give u, or the work done expansively, and so on, as in the foregoing case.

EXAMPLES ON ARTS. 45. AND 46.

Example 1. Given $P=48$, $l=2$, $h=8$, $e=0$, $n=6$; required the mean pressure of the steam. Here $a=\frac{8-2}{6}=1$.

First, by Marriotte's law.

By eq. (3), Art. 45., we have

$$u=\frac{1\times2\times48}{3}\left\{1+\frac{1}{8}+4\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)+2\left(\frac{1}{4}+\frac{1}{6}\right)\right\}$$
$$=133\cdot2,$$

which is the work done expansively.

Substituting this result in eq. (1), we have

$$u'=133\cdot2+48\times2=229\cdot2,$$

therefore by eq. (4), we have

$$P'=\frac{229\cdot2}{8}=28\cdot8 \text{ lbs.}$$

which is the mean pressure required.

Let us now calculate u by formula (2);

$$u=1\times2\times48\left\{\frac{1}{2}\left(1+\frac{1}{8}\right)+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right\}$$
$$=158\cdot7,$$

which is greater than the value derived from eq. (3); but if n were taken greater, the results by these formulæ would very nearly coincide.

Second, by the method of Art. 46.

Here by the table, when $P=48$, we find $V=573$; hence we have

$$V_1=573+\frac{1\times573}{2}=573+286=859,$$

therefore from the table $P_1=31$ nearly.

$$V_2=573+2\times286=1146,$$

therefore from the table $P_1=23$ nearly.

$$V_3=573+3\times286=1431,$$

therefore from the table, $P_3=18$ nearly.

Proceeding in this way, we find, $P_4=15$, $P_5=13$, $P_6=11$.

Substituting these values in eq. (2) Art. **33.**, observing that $\frac{s}{n}=a$, we get

$$u=\frac{1}{3}\{48+11+4(31+18+13)+2(23+15)\}=128.$$

Again we have

$$u'=128+48\times 2=224,$$

$$\text{and } P'=\frac{224}{8}=28 \text{ lbs.}$$

which nearly coincides with the value derived from Marriotte's law.

Example 2. Given $K=1440$, $l=1$, $e=\frac{1}{4}$, $h=4\frac{3}{4}$, $P=30$, elasticity vapour in condenser or $p_1=4$, $c=\cdot 2$, $f_1=\frac{1}{7}$, $f_2=1+4=5$; required L and N. Here $a=\frac{1}{4}(4\frac{3}{4}-\frac{3}{4})=1$.

Taking $n=4$, we have by eq. (3),

$$u=\frac{1\times 1\times 30}{3}\{1+\tfrac{1}{5}+4(\tfrac{1}{2}+\tfrac{1}{4})+2\times\tfrac{1}{3}\}=48\cdot 66.$$

Substituting in eq. (1), we have

$$u'=48\cdot 66+30(1-\tfrac{1}{4})=71\cdot 16,$$

and by eq. (4), we have

$$P'=\frac{71\cdot 16}{4\frac{3}{4}}=14\cdot 9 \text{ lbs.}$$

Again, by eq. (5) Art. **45.**,

$$L=\frac{71\cdot 16-4\frac{3}{4}\times 5}{4\frac{3}{4}(1+\frac{1}{7})}=8\cdot 7 \text{ lbs.}$$

Now from the table $V=882$, when $P=30$, hence we have by eq. (7), Art. **45.**,

$$N=\frac{144\times\cdot 2\times 4\frac{3}{4}\times 8\cdot 7\times 882}{33000\times 1}=32 \text{ nearly.}$$

Example 3. If $c_1=11\cdot 5$; required the duty of the engine in the last example.

Here by eq. (9), Art. **42.**, we have

$$D=\frac{11\cdot 5\times 32\times 33000}{\cdot 2}=60 \text{ millions.}$$

ACCUMULATED WORK.

47. In order to give motion to a body, there must be work done upon it; the work which a body in motion contains has been called accumulated work. Thus we may give the velocity of $32\frac{1}{6}$ feet to a body weighing w lbs. by raising it to the height of $16\frac{1}{12}$ feet, and then allowing it to fall by the force of gravity; in this case the units of work accumulated in the body will be $16\frac{1}{12} \times w$. Again, when a heavy fly wheel is in rapid motion, a considerable portion of the work of the engine, must have gone to produce this motion; and before the engine can come to a state of rest, all the work accumulated in the fly, as well as in the other parts of the machine, must be destroyed. In this way a fly wheel acts as a reservoir of work.

In order to estimate the work in a moving body, it is simply necessary to consider the height from which it must fall to acquire the given velocity, and then the work will be found, by multiplying that height in feet, by the weight of the body in lbs.; because the work expended in raising the body, to the height necessary to communicate the given velocity, must be the same as the work which gravity will perform upon the body in its descent.

Let u be put for the work accumulated in a body, whose weight is w lbs., and velocity v feet per second. Put s for the height from which the body must fall in order to acquire the given velocity v; then

$$u = ws;$$

but by eq. (5), Art. **26.**,

$$s = \frac{v^2}{2g};$$

$$\therefore\ u = \frac{wv^2}{2g} \ldots (1);$$

that is to say, THE WORK ACCUMULATED IN A MOVING BODY IS EQUAL TO THE SQUARE OF THE VELOCITY IN FEET PER SECOND, MULTIPLIED BY THE WEIGHT OF THE BODY IN *lbs.*, DIVIDED BY $2 \times 32\frac{1}{6}$.

The expression $\frac{w}{g} \cdot v^2$ is called the VIS VIVA of the body; hence it follows, that

THE ACCUMULATED WORK $= \frac{1}{2}$ THE VIS VIVA.

48. If p be put for the moving pressure, and s for the space moved; then $u = s \cdot p$, and eq. (1), Art. **47.**, becomes

$$s \cdot p = \frac{wv^2}{2g} \ldots (1);$$

$$\therefore\ s = \frac{wv^2}{2gp} \ldots (2), \text{ and } p = \frac{wv^2}{2gs} \ldots (3).$$

It may further be useful to observe, that from the equality $u = s \cdot p$, we obtain

$$p = \frac{u}{s} \ldots (4),$$

that is to say, *the moving pressure is equal to the accumulated work divided by the space through which that pressure is moved.*

49. To find the work gained or lost, as the case may be, by a body which passes from the velocity v_1 to v.

$$\text{Work in the body at first} = \frac{wv_1^2}{2g},$$

$$\text{Work in the body at last} = \frac{wv^2}{2g};$$

$$\therefore\ \text{Work gained or lost, } u = \frac{wv^2}{2g} \sim \frac{wv_1^2}{2g};$$

$$\therefore\ u = \pm\frac{w(v^2 - v_1^2)}{2g} \ldots (1);$$

Where the + or − sign is taken according as v is greater or less than v_1; this result is equivalent to eq. (10), Art. **27.**

If p be the effective moving pressure and s the space through which it is moved, then eq. (9), Art. **27.**, expresses the relation of the quantities. If p, in this equation, alternately, or periodically, accelerates and retards the motion; then $v = v_1$, that is to say, the work accumulated in this case is equal to 0. This condition is fulfilled in most combinations of machinery, such as the crank and fly wheel.

PROBLEMS ON ACCUMULATED WORK.

1. A ball weighing w lbs. revolves round a vertical axis, its distance from the axis is r feet, and it makes n revolutions per min.; required the work accumulated in the ball.

$$\text{Velocity ball per sec.} = \frac{2\pi r \times n}{60} = \frac{\pi n r}{30};$$

$$\therefore\ u = \frac{w}{2g} \cdot \frac{\pi^2 n^2 r^2}{900}.$$

2. Two equal balls are made to revolve on a vertical axis at the

distances of a and b feet from it; required the point where we may suppose the weight of the two balls collected so that the work may not be altered.

Let w=the weight of each ball; k=the distance from the axis required; and suppose them to revolve so that a point at 1 foot from the axis has a velocity of v feet per second; then

$$\text{Work acc. in the balls} = \frac{w}{2g} \cdot (av)^2 + \frac{w}{2g} \cdot (bv)^2,$$

$$\text{Work of the balls collected in } k = \frac{2w}{2g} \cdot (kv)^2,$$

$$\therefore \frac{2w}{2g} \cdot (kv)^2 = \frac{w}{2g} \cdot (av)^2 + \frac{w}{2g} \cdot (bv)^2,$$

$$\therefore k = \sqrt{\tfrac{1}{2}(a^2+b^2)}.$$

The point here determined is called THE CENTRE OF GYRATION.

3. A ball weighing w lbs. is fired from the mouth of a cannon a feet long, with the velocity of v feet per sec.; it is required to find the mean pressure, p, of the elastic vapour upon the ball.

$$\text{Work accumulated in the ball} = \frac{wv^2}{2g},$$

$$\text{Work done by the mean pressure} = ap,$$

$$\therefore ap = \frac{wv^2}{2g}, \quad \therefore p = \frac{wv^2}{2ag}.$$

4. The weight, w, of a fly wheel is collected in a point at the distance of k feet from the axis. The wheel makes n revolutions per min., the diameter of the axis is a inches, and the coefficient of friction on the axis f. How many revolutions, x, will the wheel make before it stops?

Here proceeding as in Prob. 1., we find

$$\text{Work accumulated in the wheel} = \frac{w}{2g} \cdot \frac{\pi^2 n^2 k^2}{900}.$$

$$\text{Resistance of friction} = fw,$$

$$\text{Work destroyed by friction in 1 revo.} = fw \times \frac{\pi a}{12},$$

$$\therefore \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{in } x \text{ revo.} = fw \times \frac{\pi a}{12} \times x,$$

and when the wheel stops, we have

$$fw \times \frac{\pi a}{12} \times x = \frac{w}{2g} \cdot \frac{\pi^2 n^2 k^2}{900},$$

$$\therefore \ x = \frac{\pi n^2 k^2}{150 f a}.$$

5. Required the number of strokes, x, which the fly wheel, in the last problem, will give to a forge hammer, whose weight is W lbs. and lift h feet, supposing the hammer to make one lift for every revolution of the wheel.

$$\text{Work due to raising hammer} = \mathrm{W} h x,$$

$$\text{,, \quad ,, \quad friction} = fw \times \frac{\pi a}{12} \times x;$$

but the sum of these must be equal to the whole work accumulated in the wheel;

$$\therefore \ \mathrm{W} h x + fw \times \frac{\pi a}{12} \times x = \frac{w}{2g} \cdot \frac{\pi^2 n^2 k^2}{900},$$

$$\therefore \ x = \frac{w \pi^2 n^2 k^2}{150 g (12 \mathrm{W} h + \pi a f w)}.$$

6. Two weights, P and Q, are connected by a cord that goes over a fixed pulley, C; through what space, S, must descend in order to acquire a velocity of v feet per second?

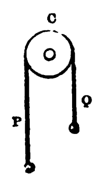


Fig. 12.

$$\text{Work accumulated in the weights} = \frac{\mathrm{P} v^2}{2g} + \frac{\mathrm{Q} v^2}{2g},$$

$$\text{Work done on P by gravity} = \mathrm{P} \cdot \mathrm{S},$$

$$\text{,, \quad ,, \quad Q \quad ,,} \quad = \mathrm{Q} \cdot \mathrm{S},$$

$$\therefore \text{ Work accumulated by gravity in the wts.} = \mathrm{P} \cdot \mathrm{S} - \mathrm{Q} \cdot \mathrm{S};$$

$$\therefore \ \mathrm{P} \cdot \mathrm{S} - \mathrm{Q} \cdot \mathrm{S} = \frac{\mathrm{P} v^2}{2g} + \frac{\mathrm{Q} v^2}{2g},$$

$$\therefore \ \mathrm{S} = \frac{v^2}{2g} \cdot \frac{\mathrm{P} + \mathrm{Q}}{\mathrm{P} - \mathrm{Q}}.$$

Motion on a horizontal plane, the resistance of friction being given.

7. A ball weighing w lbs. is projected on a horizontal plane with the velocity of v feet per second. What space, S, will the ball move over before it comes to a state of rest, allowing the coefficient of friction to be f?

Here, the retarding pressure, $p=fw$,

$$\therefore \text{ Work destroyed by friction}=fw\,s;$$

but by eq. (1), Art. 47.

$$\text{The work in the ball}=\frac{wv^2}{2g}.$$

Now when the ball stops, the work destroyed by friction must be equal to the work accumulated in the ball;

$$\therefore fw\,s=\frac{wv^2}{2g},$$

$$\therefore s=\frac{v^2}{2gf}\ldots(1).$$

8. Required the time, t, before the ball in the last problem will come to a state of rest.

Here the friction obviously acts as a uniform retarding force, and consequently the motion of the ball will be uniformly retarded. The space, s, which the ball passes over before it comes to a state of rest, will be described with the mean velocity $\frac{v}{2}$, hence we have, (see Art. 26. eq. (2)),

$$t=\frac{s}{\frac{1}{2}v}=\frac{2s}{v}.$$

Substituting the value of s given in eq. (1), Prob. 7., and reducing, we get

$$t=\frac{v}{gf}\ldots(1).$$

Or thus,

By eq. (1), Art. 27.,

$$g_1=\frac{p}{w}\cdot g=\frac{fw}{w}\cdot g=fg;$$

$$\text{but } v=g_1 t,$$

$$\therefore t=\frac{v}{g_1}=\frac{v}{fg}.$$

9. A railway train, weighing T tons, has a velocity of v_1 feet per second when the steam is turned off; what distance, s, will the train have moved over on a level rail, whose friction is p lbs. per ton, when the velocity is v feet per second?

Here, the work lost $u = p\,\text{T}\,\text{s}$,

Weight of the train in lbs. $=2240\,\text{T}$;

hence we have by eq. (1), Art. 49.,

$$p\,\text{T}\,\text{s}=\frac{2240\,\text{T}(v_1^2-v^2)}{2g};$$

$$\therefore\ \text{s}=\frac{1120(v_1^2-v^2)}{gp}\ldots(1),$$

when the train comes to a state of rest $v=0$, and then the whole space moved over will be

$$\text{s}=\frac{1120v_1^2}{gp}\ldots(2).$$

10. To find the time, t, at which the train in the last problem will have the velocity v.

Here the mean velocity, with which the space s is described, is $\frac{v_1+v}{2}$;

$$\therefore\ t=\text{s}\div\frac{v_1+v}{2}=\frac{2\text{s}}{v_1+v}.$$

Substituting the value of s given in eq. (1), Prob. 9., and reducing, we get

$$t=\frac{2240(v_1-v)}{pg}.$$

Or thus,

Retarding pressure, $=p\,\text{T}$, and weight mass moved $=2240\,\text{T}$; hence we have by eq. (1), Art. 27.,

$$g_1=\frac{p\,\text{T}}{2240\,\text{T}}\cdot g=\frac{pg}{2240},$$

and by eq. (4), Art. 27., we get

$$t=\frac{v_1-v}{g_1}=\frac{2240(v_1-v)}{pg}.$$

When $v=0$, the time which the train takes to come to rest is $\frac{2240v_1}{pg}$.

11. A pressure, P, acts upon a body parallel to the plane; required the space, s, moved over when the body has attained a given velocity, v.

Work in moving over s ft. $=\text{P}\,.\,\text{s}-fw\,.\,\text{s}$, . . . (1);

$$\text{Work accumulated} = \frac{wv^2}{2g};$$

$$\therefore \text{P} . \text{S} - fw . \text{S} = \frac{wv^2}{2g},$$

$$\therefore \text{S} = \frac{wv^2}{2g(\text{P} - fw)} \quad . \; . \; . \; (2).$$

12. Suppose the body in the last problem to be moved for t seconds; required the velocity, v, acquired, and the work, u, accumulated.

By eq. (2), Art. 26., we have

$$\text{S} = \frac{v}{2} . t,$$

but the value of S is also given in eq. (2) of the last problem,

$$\therefore \frac{v}{2} . t = \frac{wv^2}{2g(\text{P} - fw)},$$

$$\therefore v = \frac{\text{P} - fw}{w} . tg.$$

Or thus,

Dividing eq. (1), of the last problem, by S, we get

the moving pressure$=\text{P}-fw$;

hence we have by eq. (1) Art. 27.,

$$g_1 = \frac{p}{w} . g = \frac{\text{P} - fw}{w} . g;$$

$$\text{but } v = tg_1$$

$$= \frac{\text{P} - fw}{w} . tg \quad . \; . \; . \; (1).$$

Now $u = \frac{wv^2}{2g}$; hence we have by substituting the value of v, and reducing,

$$u = \frac{(\text{P} - fw)^2 t^2 g}{2w} \quad . \; . \; . \; (2).$$

Motion of bodies on an inclined plane, when its height is small as composed with its length.

13. A train of T tons descends an incline of S feet in length, having a total rise of h feet; what will be the velocity, v, acquired by the train, supposing the friction to be p lbs. per ton?

Work done on the train = work gravity − work friction

$$=2240 \times \text{T} \times h - p \times \text{T} \times \text{S};$$

but we also have

$$\text{Work accumulated in the train} = \frac{v^2 \times \text{T} \times 2240}{2g},$$

$$\therefore \frac{v^2 \times \text{T} \times 2240}{2g} = 2240 \times \text{T} \times h - p \times \text{T} \times \text{S},$$

$$\therefore v = \sqrt{2gh - \frac{1}{120} \cdot gp\text{S}}.$$

14. If the velocity of the train, in the last problem, be v_1 feet per second when the steam is turned off; what will be its velocity, v, when it arrives at the bottom of the incline?

Work accumulated in the train when the steam is turned off

$$= \frac{v_1^2 \times \text{T} \times 2240}{2g};$$

Work done on the train in descending $= 2240\,\text{T}h - p\text{TS}$;

$$\text{Total work accumulated in the train} = \frac{v^2 \times \text{T} \times 2240}{2g};$$

$$\therefore \frac{v_1^2 \times \text{T} \times 2240}{2g} + 2240\,\text{T}h - p\text{TS} = \frac{v^2 \times \text{T} \times 2240}{2g},$$

$$\therefore v = \sqrt{v_1^2 + 2g\left(h - \frac{1}{2240} \cdot p\text{S}\right)}.$$

15. How far will the train in the last problem, move along a horizontal plane?

Let x = the distance; then

Work destroyed on the horizontal plane $= p\text{T}x$;

but this must be equal to the work accumulated in the train,

$$\therefore p\text{T}x = \frac{v^2 \times \text{T} \times 2240}{2g},$$

$$x = \frac{1120 v^2}{gp},$$

where v is determined in Problem 14.

16. If the train ascend an incline, having a rise of e feet in 100 feet, with the velocity v_1 feet per sec. when the steam is turned off; through what space, x, will it move before it comes to a state of rest?

$$\text{Rise of the rail in } x \text{ ft.} = \frac{ex}{100};$$

F

$$\therefore \text{ Work destroyed} = \text{T} \times 2240 \times \frac{ex}{100} + p\text{T}x;$$

but this must be equal to the work accumulated in the train;

$$\therefore \text{T} \times 22\cdot4 \times ex + p\text{T}x = \frac{v_1^2 \times \text{T} \times 2240}{2g},$$

$$\therefore x = \frac{1120 v_1^2}{g(22\cdot4\, e + p)}.$$

17. Suppose the train, in Problem 13., to be attached to a rope, passing round a wheel at the top of the incline, which has an empty train of T_1 tons attached to the other extremity of the rope: what velocity, v, will the train acquire in descending s feet of the incline?

Work done on the descending train $= \text{T}(2240h - p\text{s})$;

Work expended on the ascending train $= \text{T}_1(2240h + p\text{s})$;

now the difference between these must be equal to the work accumulated in the two trains;

$$\therefore \text{T}(2240h - p\text{s}) - \text{T}_1(2240h + p\text{s}) = \frac{v^2 \times (\text{T} + \text{T}_1)2240}{2g},$$

$$\therefore v = \sqrt{2g\left(h \cdot \frac{\text{T} - \text{T}_1}{\text{T} + \text{T}_1} - \frac{p\text{s}}{2240}\right)}.$$

PART II.

STATICS.

CHAP. III.

STATICAL FORCES. PARALLELOGRAM OF FORCES.

50. When two or more forces act at one time upon a particle, such that they do not destroy one another, it will move or tend to move in some particular direction, and with a definite force. Now the forces actually applied to the particle are called the *component* forces, and the single force which is produced by their combined action is called the *resultant.*

Axioms.

1. When forces act in the same straight line, and in the same direction, the resultant is equal to the sum of the component forces; thus if a body be acted upon in the same direction by two pressures, one of 2 lbs. and the other 3 lbs., then the resultant pressure will be 5 lbs., that is to say, a single pressure of 5 lbs. will produce the same effect as the other two pressures acting separately. On the contrary, when some of the forces act in the contrary direction, the resultant is equal to the sum of the forces acting in one direction diminished by the sum of those acting in the opposite direction: thus if a body be acted upon in contrary directions by two pressures, one of 8 lbs. and the other 6 lbs., then the resultant pressure will be 2 lbs.

If the forces which act in one direction be considered positive, and those in the other direction negative, the single or resultant force is equal to the algebraic sum of the components.

Let a point A be acted upon by two forces P and P_1 tending to move it in the same direction, viz. towards E. Take AP equal to the units of pressure in P,

Fig. 13.

and A P_1 equal to the units in P_1, then the units in the resultant $= P + P_1 = AP + AP_1$.

Now if P_1 acts in a contrary direction to P, the units in P_1 or A P_1 must be taken negatively, that is A P_1 must be measured in a direction contrary to A P, in this case, therefore, the number of units in the resultant is equal to the algebraic sum of the lines representing the component forces.

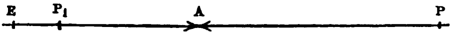


Fig. 14.

2. Two equal forces, acting in opposite directions, destroy each other.

Hence we may add equal and opposite forces without affecting the equilibrium of a system of forces. And in like manner we may remove equal and opposite forces.

3. A force may act on a rigid body at any point in the line of its direction, without altering the effect.

The truth of this axiom depends upon observation and experiment. Thus a force P acting on a rigid surface, C D (supposed to be without weight) may be applied at any point, A or B, in the line of its direction A P without altering the effect. Thus, if a weight be suspended by a string, the tension of the string, or the force with which it is pulled, is everywhere the same, whatever may be its length. Thus the same force is required to push or to pull a boat in the direction of its length; and the effect is the same, whether the steam carriage pulls or pushes the railway train forward.

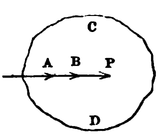


Fig. 15.

4. The resultant of two equal forces bisects the angle between their directions. For no reason can be assigned why the resultant should lie more towards one force than towards the other. In this case *the direction* of the resultant is obviously in the diagonal, A R, of the parallelogram A B R D, constructed upon the lines A B and A D, representing the directions and magnitudes of the two equal forces.

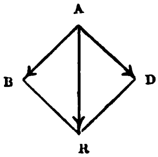


Fig 16.

5. The resultant of two forces passes through the point where the directions of these forces intersect; and conversely the directions of the two component forces pass through a point lying in the direction of their resultant.

6. If a uniform heavy bar be suspended by its middle point, it will hang in a horizontal position. For if the bar be placed in a horizontal position, no reason can be assigned why one extremity should preponderate more than the other.

The tension of the string from which the bar is suspended will be in proportion to the length of the bar.

We proceed now to demonstrate the fundamental proposition relative to the resultant of parallel forees, as well as that of oblique forces.*

Resultant of Parallel Forces.

PROP. 1.—*The resultant of two parallel forces,* P *and* Q, *is equal to the sum of the two forces, and acts parallel to their direction, and also divides the line connecting their points of application in such a manner that the two component forces are inversely as their distance from this point.*

51. Thus let the two parallel forces P and Q be applied (see *fig.* 18.) at the points A and B of a rigid surface KL without weight; also let C be the point at which the resultant P+Q acts; then P : Q :: BC : AC. See also *fig.* 17.

Let JE and ED be two horizontal bars, or uniform rods, suspended from their middle points A and B (see Axiom 6.); and let P be the weight of the bar JE, and Q the weight of ED; then the bar JE will produce a force in the vertical direction AL equal to P, and the bar ED a force in the vertical BN equal to Q. Now let the two bars be connected together at E, —this will obviously not at all affect the state of equilibrium. Let C be the middle of the whole bar JD,

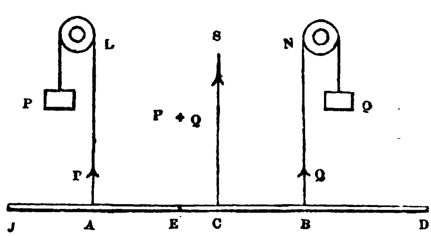


Fig. 17.

and suppose a string CS to be attached to this point. Now, if we suppose the strings AL and BN to be cut, the bar JD would remain suspended by the string CS, and the force exerted upon it would be equal to P+Q, the weight of the whole bar. Hence it appears that the forces P and Q acting together are equivalent to a single force equal to P+Q, acting at the point C, or in other words the resultant of P and Q is P+Q acting at the point C.

* The following proof of the parallelogram of forces, was first given by the Author in the "English Journal of Education."

To find the relation of AC and BC, we have

$$P : Q :: JE : ED,$$

$$\therefore \frac{P}{Q} = \frac{JE}{ED} = \frac{JD - DE}{JD - JE} = \frac{2DC - 2DB}{2JC - 2JA} = \frac{DC - DB}{JC - JA} = \frac{BC}{AC}.$$

which is the proposition enunciated, when the forces act perpendicular to the line AB, joining their points of application.

Let the parallel forces P and Q act in the plane rigid surface KL, which is supposed to be without weight, in the directions AP and B^1Q; and let AB^1 be drawn perpendicular to AP or B^1Q; then, supposing A and B^1 to be the points at which the forces P and Q are applied, and C^1 the point through which their resultant acts, we have by the foregoing result,

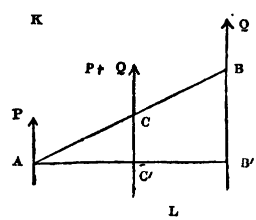


Fig. 18.

$$\frac{P}{Q} = \frac{B^1C^1}{AC^1}.$$

From A draw any straight line AB cutting the direction of the force Q in B, and that of the resultant in C.

Now, by Axiom 3., any one of these forces may act at any point of the line of its direction without altering the effect. Hence, therefore, we may suppose Q to act at B and the resultant at C.

Since CC^1 and BB^1 are parallel, we have on the principle of similar triangles,

$$\frac{BC}{AC} = \frac{B^1C^1}{AC^1},$$

$$\therefore \frac{P}{Q} = \frac{BC}{AC},$$

$$\text{or } P : Q :: BC : AC,$$

which is the general form of the proposition.

Resultant of Oblique Forces. The parallelogram of forces.

52. The parallelogram of forces is this; if the sides AD and AB (see *fig.* 16.) of the parallelogram ABRD represent (see Art. **21.**) two forces acting on the point A, then the diagonal AR will represent the resultant. We shall first prove that the diagonal is the *direction* of the resultant, and second that it represents the resultant in *magnitude*.

PROP. 2.—*The resultant of two forces acting on the same point is in the* DIRECTION *of the diagonal of the parallelogram described on the two sides representing the magnitude and direction of the two component forces.*

53. Let AE and BL represent two parallel forces, P and Q, acting at the points A and B, of a rigid surface, and SCD the direction of their resultant. In the line AB produced, let two equal forces, represented by AK and BF, each equal to Q, act in contrary directions. These two forces, by Axiom 2., will not alter the direction SCD of the resultant of the parallel forces P and Q, therefore SCD will be the direction of the resultant of all the forces now applied. Describe the parallelogram BFNL; draw the diagonal BN, and produce it until it intersects SCD in D; then, by Axiom 4., NBD will be the direction of the resultant of the equal forces BF and BL. Join DA and produce it to R; then DR will be the direction of the resultant of the forces AK and AE; for since SCD is the direction of the resultant of all the forces applied, and ND is the direction of the resultant of BF and BL, it follows, by Axiom 5., that the resultant of AK and AE must pass through the point D. Draw KR parallel to AE, and join RE.

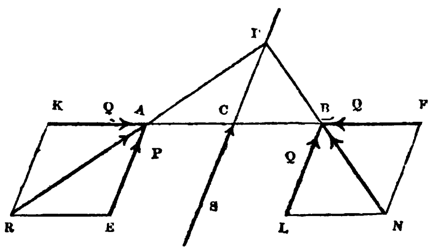


Fig. 19.

By geometry,

$$\angle CDB = \angle LBN, \text{ and } \angle CBD = \angle FBN,$$

$$\text{but } \angle LBN = \angle FBN, \ \therefore \ \angle CDB = \angle CBD;$$

$$\therefore \ CD = BC;$$

by the similar triangles AKR and ADC,

$$\frac{KR}{AK} = \frac{CD}{AC} = \frac{BC}{AC}.$$

Now, by Prop. 1., we have

$$\frac{P}{Q} = \frac{BC}{AC},$$

$$\therefore \ \frac{P}{Q} = \frac{KR}{AK},$$

$$\text{but } AK = Q, \ \therefore \ KR = P = AE;$$

consequently AKRE is a parallelogram described on the sides AE and AK, representing the forces P and Q respectively, and the diagonal AR is the direction of their resultant.

PROP. 3.—*The resultant of two forces acting on the same point is represented in* DIRECTION *and* MAGNITUDE *by the diagonal of the parallelogram described on the two sides representing the magnitude and direction of the two component forces.*

54. Let AF and AB represent the two component forces, P and Q, acting on the point A, and let AD represent a third force R, which, acting at A, maintains P and Q in equilibrium; then this force, R, must be equal to the resultant of P and Q, and must act in a direction contrary to it. Moreover, the resultant of Q and R must be equal and contrary to P. Complete the parallelograms ABEF and ABCD; then, since (by prop. 2.) AE is the direction of the resultant of P and Q, it must be in the same direction as AD, that is, AD and AE are in the same straight line; and for the same reason, AC and AF are also in the same straight line; hence ACBE is a parallelogram, and therefore AE=CB=AD=R, that is to say, the diagonal AE represents the magnitude of the resultant of the forces P and Q.

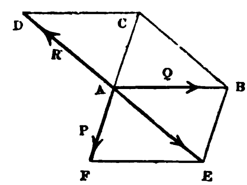


Fig. 20.

We shall now give Duchayla's proof of this important proposition.

Duchayla's proof of the Parallelogram of Forces.

55. First, to prove that the diagonal of the parallelogram is *the direction* of the resultant.

Suppose the proposition to be true for two forces P and P_1, and also for P and P_2; then we shall show that it will be true for P and P_1+P_2.

Let A be the point at which the forces are applied, viz., P acting in the direction AF, and P_1+P_2 in the direction AC.

Take AF equal to the units of pressure in the force P, AB equal to P_1, and BC equal to P_2; and complete the parallelograms AFEB and BCDE.

We may suppose (Axiom 3.) P_1 to be applied at A and P_2 at B. By hypothesis the resultant of P and P_1 applied at A is in the direction AE, but these two forces may act at E in the directions BE and FE respectively*, that is to say, P in BE and P_1 in FE; now as forces may act in any line of their direction, P may act at B in the direction BE, and P_1 at D in the direction FED.

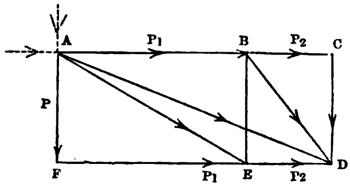


Fig. 21.

By hypothesis BD is the direction of the resultant of P and P_2 applied at B in the directions BE and BC respectively; these two forces may be applied at D in the directions CD and ED respectively, that is to say, P in CD and P_2 in ED.

Now we have transferred all our forces from A to D, that is, P originally acting at A in the direction AF to D acting in the parallel direction CD, and P_1+P_2 originally acting at A in the direction AC to D acting in the parallel direction FD; consequently D must be a point in the line of the resultant's action, and therefore AD is the direction of the resultant. We have therefore proved, that if the proposition be true for the forces P and P_1, and also for P and P_2, it is true for P and P_1+P_2.

But, by Axiom 4., it is true for any equal pair of forces P and P, and also for another pair P and P, therefore by the preceding investigation it is true for P and 2P; and since it is true for P and P, and also for P and 2P, it is true for P and 3P; and so on, it is true for P and nP. Again, since it is true for nP and P, and also for nP and P, it is true for nP and 2P, and so on generally for nP and mP, where n and m are any whole number. But as P may be any whole or fraction quantity, the proposition is true for any commensurate forces whatever.

* Suppose two forces P and P_1 to be applied to the point A of a rigid surface; now as these forces must have a resultant, let AE be the direction of this resultant, and since it may be applied at any point of its direction, we may suppose it to be applied at E; but this resultant would be produced by the forces P and P_1 acting at E respectively parallel to their original directions; hence it appears that we may transfer the two forces P and P_1 to any point E in the line of their resultant without altering their effect. The converse of this proposition also holds good.

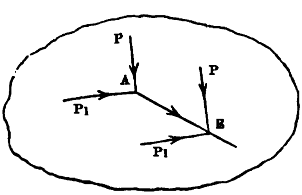


Fig. 22.

We now proceed to show that the proposition extends to incommensurable forces. Let AB and AF represent two such forces acting at A; complete the parallelogram ABEF, and if AE is not the direction of the resultant let it be in some other direction AJ. Suppose AB to be divided into a number of equal parts, each less than JE, and let these divisions be marked off along BE, then one of these divisions must fall between J and E, suppose at C; complete the parallelogram ABCD, then AC is the resultant of the commensurable forces AB and AD; but AC lies further away from AB than AJ does, although the component force AD is less than AF, which is absurd; therefore AJ is not the direction of the resultant, and it may in like manner be shown that no other line, except AE is in that direction. Hence the proposition is true as well for incommensurable forces as for commensurable ones.

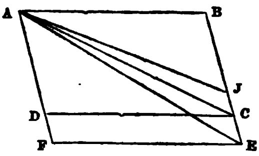


Fig. 23.

Secondly, to prove that the diagonal of the parallelogram represents the *magnitude* of the resultant when the sides are respectively taken to represent the component forces. See Prop. 3. Art. 54.

APPLICATIONS.

56. By the parallelogram of forces we are enabled to *compound* two forces into one, and also conversely to *resolve* any single force into two, or, in other words, to find two forces which shall produce precisely the same effect as the given force. The former operation is called the *composition* of forces, the latter one the *resolution* of forces.

57. Before proceeding to any other general investigations, we shall take a few examples of the parallelogram of forces.

Ex. 1. Two forces, equal to 6 lbs. and 8 lbs. respectively, act upon a particle A, at right angles to each other; required to find their resultant.

Draw AC and AB at right angles to each other; from a scale of equal parts take AC=6, and AB=8; complete the rectangle ABRC; then AR will represent the direction and magnitude of the resultant.

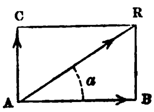


Fig. 24.

From the right-angled triangle ABR, we have

$$AR=\sqrt{AB^2+BR^2}=\sqrt{8^2+6^2}=10,$$

that is, the force of the resultant is measured by 10 lbs.

Ex. 2. In pulling a tree, AB, down by means of a rope CA, a force of 6 cwt. is applied to the rope; it is required to find the effective force tending to draw the tree over, when the horizontal distance CB=30 ft., and AB=40 ft.

Here the effective force must act perpendicularly to AB. Take AR=6, and complete the rectangle AFRE; then the force represented by AR will be resolved into the two component forces represented by AE and AF, the former acts in the direction of the length of the tree, and therefore has no tendency to pull it over; but the latter acts perpendicular to the length of the tree, and is therefore the effective force tending to pull it over.

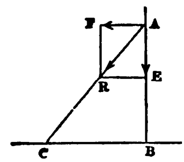


Fig. 25.

To find AF or RE, we have from the similar triangles ARE and ACB

$$\text{RE}=\frac{\text{CB}\cdot\text{AR}}{\text{AC}},$$

but CB=30, AB=40, AR=6, and $\text{AC}=\sqrt{\text{CB}^2+\text{AB}^2}=50$,

$$\therefore\ \text{RE}=\frac{30\times 6}{50}=3{\cdot}6,$$

that is, the effective force is 3·6 cwts.

Ex. 3. Required the effective force, P, as in the last example, when $\angle\,\text{CAB}=\theta$, and R is the force applied to the rope AC.

Here by trigonometry,

$$\text{ER or AF}=\text{AR}\sin\text{CAB, that is,}$$

$$\text{P}=\text{R}\sin\theta\ \ldots\ (1).$$

If $\theta=30°$, then $\sin\theta=\frac{1}{2}$, and

$$\therefore\ \text{P}=\tfrac{1}{2}\text{R},$$

that is to say, the effective force in this case is only one-half the force applied.

The Triangle of Forces.

58. If three forces, P, Q, and R, act upon a point A in the directions AC, AD, BA, and keep it in equilibrium, the sides of the triangle ABC, formed by drawing BC parallel to AD, will respectively represent the three forces. For if AC and AD represent P and Q respectively, the diagonal AB, of the parallelogram ADBC, will represent R; but CB is equal to AD, therefore the three forces P, Q, and R, will be respectively represented by AC, CB, and AB. This proposition is called *the triangle of forces.*

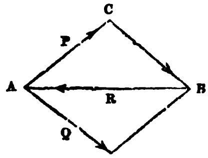


Fig. 26.

It follows from the triangle of forces, that any properties relative to the sides and angles of a triangle, applies to the magnitude and directions of three forces in equilibrium.

Thus if $\theta = \angle CAD$ the angle between the directions of the forces; then, from the triangle ABC, we have by trigonometry

$$AB^2 = AC^2 + BC^2 - 2\ AC \cdot BC \cdot \cos ACB,$$

but $\cos ACB = -\cos CAD = -\cos \theta$,

$$\therefore\ R^2 = P^2 + Q^2 + 2\ PQ \cos \theta \ \ldots \ (1).$$

Polygon of Forces.

59. In like manner it may be shown, that if any number of forces, acting on a point, are in equilibrium, they may be respectively represented by the sides of a polygon, formed by lines parallel to the directions in which the forces act.

Let AP_1, AP_2, AP_3, AP_4, AP_5, represent the forces P_1, P_2, P_3, P_4, P_5, acting at the point A, as in the figure. Complete the parallelograms AP_1CP_2, $ACDP_3$, $ADEP_4$, drawing the diagonals AC, AD, AE; then AC is the resultant of P_1 and P_2, AD of AC and P_3, and AE of AD and P_4; and since the forces are in equilibrium AE is equal and opposite to P_5; therefore the sides of the polygon AP_1CDE respectively represent in order the forces in direction and magnitude, that is, P_1 is represented by AP_1, P_2 by P_1C, P_3 by CD, P_4 by DE, and P_5 by AE. It will be observed from the foregoing demonstration, that it is not necessary the forces should lie in the same plane. This proposition is called *the polygon of forces.*

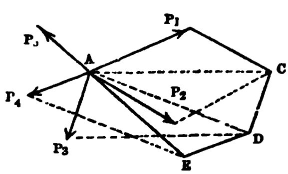


Fig. 27.

Parallelopiped of Forces.

60. Let the three edges AB, AC, and AD of a parallelopiped represent three forces applied at A, then the resultant of these three forces will be represented by the diagonal AF.

The resultant of the forces AC and AB is AE the diagonal of the face ACEB; and compounding this with the force AD, we have AF, the diagonal of the parallelogram ADFE, for the resultant.

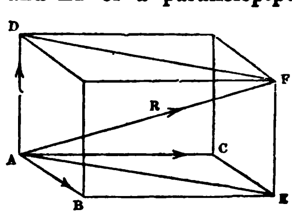


Fig. 28.

Exercises for the Student.

1. Three forces of 5 lbs., 3 lbs., and 2 lbs. respectively, act upon a point in the same direction, and in the opposite direction two other forces of 8 lbs. and 9 lbs. act; what single force will keep the point at rest? *Ans.* 7 lbs.

2. Two forces, of $5\frac{1}{2}$ and $3\frac{1}{2}$ lbs., applied at a point, have a resultant of 9 lbs.; in what direction do the forces act? *Ans.* In the same straight line.

3. Two forces, whose magnitudes are as 3 to 4, acting on a point at right angles to each other, produce a resultant of 20 lbs.; required the component forces? *Ans.* 12 lbs. and 16 lbs.

4. Two forces, of 9 lbs. and 12 lbs., act at right angles on a point; required the magnitude of their resultant. *Ans.* 15 lbs.

5. If three forces acting on a point keep it at rest, when their intensities are doubled they will still keep it at rest if their directions be not changed.

6. Let ABC be a triangle, and D the middle point of the side BC. If the three forces represented in magnitude and direction by AB, AC, and AD act upon the point A; find the direction and magnitude of the resultant. *Ans.* The direction is in the line AD, and the magnitude represented by 3 AD.

7. If two equal forces, P, act at right angles to each other, show that their resultant $= P\sqrt{2}$.

8. If two equal forces meet at an angle of 120°, show that their resultant is equal to either of them.

9. What is the magnitude of the resultant in eq. (1), Art. 58., when $\theta = 0$? *Ans.* $P+Q$.

10. If $3p$ and $4p$ be the magnitudes of two component forces acting at right angles to each other, then $5p$ will be the magnitude of the resultant.

11. If two equal forces, P, meet at an angle θ, show that their resultant $R = 2P \cos\frac{\theta}{2}$.

12. A boat is fastened to a fixed point by a rope, and is at the same time acted on by the wind, with a force P, and by the current with a force Q; required the tension of the cord when the two forces act at right angles to each other. *Ans.* $\sqrt{P^2+Q^2}$.

CHAP. IV.

GENERAL PROPOSITIONS RELATIVE TO FORCES. WORK OF FORCES, ETC.

FORCES ACTING IN THE SAME PLANE.

61. ANY number of forces acting on a rigid surface may all be transferred to one point without altering their effect.

Let the forces P_1, P_2, P_3, P_4, be applied respectively to the points a, b, c, d, in a plane rigid surface. Let A be the intersection of P_1 and P_2, and let AB be the direction of their resultant R_1; let B be the intersection of R_1 and P_3, and let BC be the direction of their resultant R_2; let C be the intersection of R_2 and P_4, and let

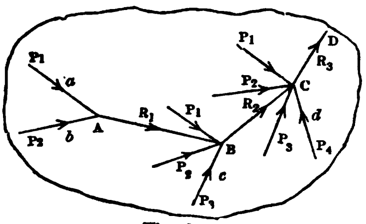


Fig. 29.

CD be the direction of their resultant; and so on to any number of forces. Now since AB is the direction of the resultant of P_1 and P_2, they may be transferred from the point A to B (Art. 55., foot note); in like manner, since BC is the direction of the resultant of R_1 and P_3, or what is the same thing, of P_1, P_2, P_3, they may be transferred from B to C; thus all the forces are transferred to the point C acting respectively parallel to their original directions. In the same manner R_3 may be combined with P_5, R_4 with P_6, and so on to any number of forces. From this proposition it follows, that any formulæ which may hereafter be established relative to forces acting at a point in a plane, will also apply to forces acting at any points in a plane.

62. To resolve a given force P into two component forces X and Y acting at right angles to each other, so that the direction of X shall form a given angle, α, with the direction of P.

Let AR represent P; from A draw AB making the angle RAB equal to α; draw AC perpendicular to AB; from R draw RB parallel to AC, and RC parallel to AB; then from the parallelogram of forces, AB will represent X, and AC will represent Y.

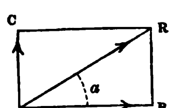


Fig. 30.

Hence, by trigonometry, we have

$$AB = AR \times \cos\alpha, \text{ and } AC = AR \times \sin\alpha,$$

$$\therefore X = P\cos\alpha, \text{ and } Y = P\sin\alpha.$$

Moreover we also have

$$\tan\alpha = \frac{Y}{X}, \text{ and } X^2 + Y^2 = P^2.$$

63. Any number of forces, represented by P_1, P_2, P_3, &c., in the same plane act at the point A; to find the direction and magnitude of their resultant R.

Through A draw any rectangular axes Ax and Ay; complete the parallelograms $AX_1P_1Y_1$, $AX_2P_2Y_2$, &c. By this construction each force is resolved into two component forces, one acting along the axis Ax and the other along Ay; thus P_1 is resolved into AX_1 and AY_1, and so on to the others. Let AX be the *algebraic sum* of the resolved forces acting along the axis Ax, and AY the sum of the resolved forces acting along Ay; complete the parallelogram AYRX; then AR is the resultant of the whole system of forces.

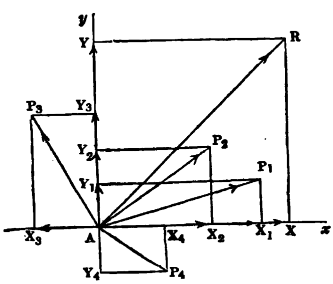


Fig. 31.

Put α_1, α_2, &c., for the angles which the direction of the forces P_1, P_2, &c., respectively make with Ax, that is, $\angle P_1Ax = \alpha_1$, $\angle P_2Ax = \alpha_2$, $\angle P_3Ax = \alpha_3$, and so on; also put $X = AX$, $Y = AY$, $R = AR$, and $\theta = \angle RAx$; then we have, observing that the forces which act in one certain direction are positive, while those which act in the contrary direction are negative,

$$X = AX_1 + AX_2 - AX_3 + AX_4 \ldots \&c.,$$

$$Y = AY_1 + AY_2 + AY_3 - AY_4 \ldots \&c.;$$

but by Art. **59.**, $AX_1 = P_1\cos\alpha_1$, $AX_2 = P_2\cos\alpha_2$, $-AX_3 = P_3\cos\alpha_3$ (observing that α_3 is an angle greater than 90° and less than 180°), and so on; also $AY_1 = P_1\sin\alpha_1$, $AY_2 = P_2\sin\alpha_2$, $AY_3 = P_3\sin\alpha_3$, $-AY_4 = P_4\sin\alpha_4$, and so on; substituting these values, we have

$$X = P_1\cos\alpha_1 + P_2\cos\alpha_2 + P_3\cos\alpha_3 + \&c. \ldots (1),$$

$$Y = P_1\sin\alpha_1 + P_2\sin\alpha_2 + P_3\sin\alpha_3 + \&c. \ldots (2).$$

The two components of the resultant R being determined by (1)

and (2), the magnitude and inclination of this resultant may be found from the right angled triangle AXR, thus

$$R^2 = X^2 + Y^2 \ldots (3),$$

$$\tan\theta = \frac{Y}{X} \ldots (4).$$

The eqs. (1), (2), (3), (4) determine all the conditions of the problem.

64. If the forces P_1, P_2, &c., mutually destroy one another, that is, if they are in equilibrium, then AX or $X=0$, and AY or $Y=0$, and hence eqs. (1) and (2) become

$$P_1 \cos\alpha_1 + P_2 \cos\alpha_2 + \&c. = 0 \ldots (5),$$

$$P_1 \sin\alpha_1 + P_2 \sin\alpha_2 + \&c. = 0 \ldots (6).$$

65. Let there be three given pressures P_1, P_2, P_3 to determine an expression for the resultant.

Squaring eqs. (1) and (2), and substituting in eq. (3), observing that $\sin^2\alpha + \cos^2\alpha = 1$, we find

$$R^2 = P_1^2 + P_2^2 + P_3^2 + 2P_1P_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$$
$$+ 2P_1P_3(\cos\alpha_1\cos\alpha_3 + \sin\alpha_1\sin\alpha_3) + 2P_2P_3(\cos\alpha_2\cos\alpha_3 + \sin\alpha_2\sin\alpha_3)$$
$$= P_1^2 + P_2^2 + P_3^2 + 2P_1P_2\cos(\alpha_1 - \alpha_1) + 2P_1P_3\cos(\alpha_1 - \alpha_2)$$
$$+ 2P_2P_3\cos(\alpha_2 - \alpha_3) \ldots (7)$$

where it will be observed that $\alpha_1 - \alpha_2$ is the angle which P_1 makes with P_2, and so on to the others. This is only a particular case of a general formula applying to any number of pressures.

If $P_3 = 0$, then this expression becomes the same as that given in eq. (1) Art. **58.**

THE EQUALITY OF MOMENTS.

66. DEFINITION. The *moment* of a force about a given point is the product of the force by the perpendicular let fall from the given point upon the direction of the force. This moment, as we shall shortly prove, measures the tendency of the force to turn the body round the given point as a fixed axis; and hence if the moments which tend to turn the body from right to left be called positive, those which tend to turn it in the contrary direction must be taken negative, and therefore the algebraic sum of a series of moments must be understood in this sense.

The principle of THE EQUALITY OF MOMENTS is this:

The sum of the moments of any number of forces acting on a rigid body is equal to the moment of the resultant of these forces.

Or, when equilibrium takes place, the proposition may be stated as follows:

The sum of the moments tending to turn a rigid body in one direction, is equal to the sum of the moments tending to turn it in the opposite direction.

Let the forces P_1 and P_2 acting at A be represented by AP_1 and AP_2, and their resultant R by AR. Take any point O as the axis of moments; draw Ay perpendicular to AOx; complete the parallelograms as in Art. 63.; and from O draw Op_1, Or, Op_2 respectively perpendicular to the directions of the forces P_1, R, P_2; then by Art. 63.,

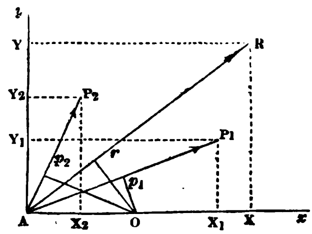


Fig. 32.

$$AY = AY_1 + AY_2 \ldots (1):$$

by the similar triangles AOr and ARY

$$AY = AR \times \frac{Or}{AO},$$

by the similar triangles AOp_1 and AP_1Y_1

$$AY_1 = AP_1 \times \frac{Op_1}{AO},$$

and by the similar triangles AOp_2 and AP_2Y_2

$$AY_2 = AP_2 \times \frac{Op_2}{AO},$$

substituting these values in eq. (1), and multiplying by the common divisor AO, we have

$$AR \times Or = AP_1 \times Op_1 + AP_2 \times Op_2,$$

or putting $R = AR$, $r = Or$, $P_1 = AP_1$, $p_1 = Op_1$, and so on, we have

$$R \cdot r = P_1 \cdot p_1 + P_2 \cdot p_2 \ldots (2),$$

but $R \cdot r$ is the moment of R, and $P_1 \cdot p_1$ is the moment of P_1, and so on,

$$\therefore \text{ moment } R = \text{moment } P_1 + \text{moment } P_2.$$

We have here taken O, the centre of moments *without* the angle formed by P_1 and P_2, now we shall take it *within* this angle, as in the annexed figure. In this case

$$AY = AY_1 - AY_2,$$

and, proceeding as before, we find

$$R \cdot r = P_1 \cdot p_1 - P_2 \cdot p_2.$$

Here the moment of P_2 is negative, because it tends to turn the system about O in a direction contrary to the other forces. Hence the proposition is true as applied to two forces.

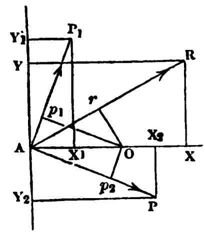


Fig. 33.

If in the last case, P_1 and P_2 are in equilibrium, $R \cdot r = 0$, and

$$P_1 \cdot p_1 = P_2 \cdot p_2 \ldots (3),$$

that is, in this case, the moment of the one force is equal to the moment of the other.

Let P_1, P_2, P_3 be three forces acting at any points in a plane rigid surface, and let any point O be taken as the axis of moments. The forces P_1 and P_2 will have a resultant R_1, in the direction AR_1 (see Art. **54.**), which being produced will intersect the direction of P_3 in a point B; now R_1 and P_3 acting at B will have a resultant R_2 acting through the same point; therefore by eq. (2), Art. **66.**,

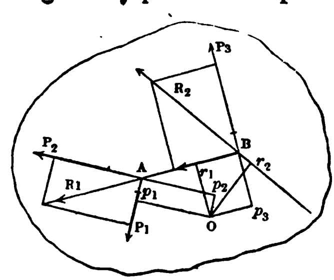


Fig. 34.

$$R_1 \cdot r_1 = P_1 \cdot p_1 + P_2 \cdot p_2,$$

and in like manner, because R_2 is the resultant of R_1 and P_3,

$$R_2 \cdot r_2 = R_1 \cdot r_1 + P_3 \cdot p_3,$$

therefore by substitution,

$$R \cdot r_2 = P_1 \cdot p_1 + P_2 \cdot p_2 + P_3 \cdot p_3.$$

In like manner R_3 will be the resultant of R_2 and P_4,

$$\therefore R_3 \cdot r_3 = R_2 \cdot r_2 + P_4 \cdot p_4,$$

and sustituting as before, we have

$$R_3 \cdot r_3 = P_1 \cdot p_1 + P_2 \cdot p_2 + P_3 \cdot p_3 + P_4 \cdot p_4,$$

and so on generally, hence we have, putting R for the last resultant,

$$R \cdot r = P_1 \cdot p_1 + P_2 \cdot p_2 + P_3 \cdot p_3 + \&c. \ldots (4);$$

but $R \cdot r$ is the moment of the resultant R, and $P_1 \cdot p_1$ is the resultant of the force P_1, and so on,

$$\therefore \text{moment } R = \text{moment } P_1 + \text{moment } P_2 + \&c.$$

It further appears, from Art. **61.**, that eqs. (1), (2), (3), and (4), of Art. **63.**, also apply to the present case.

67. If the forces are in equilibrium, then $R \cdot r=0$, and

$$P_1 \cdot p_1+P_2 \cdot p_2+P_3 \cdot p_3+\&c.=0 \ldots (5).$$

Here some of the moments must be negative; hence by transposing these negative moments, we shall have the sum of the moments turning the body in one direction equal to the sum of the moments turning it in the opposite direction.

68. If the point O, forming the centre of moments, be taken any where in the line of action of the resultant, then $r=0$, and eq. (4) becomes the same as eq. (5), which has just been found from other considerations.

When the forces acting upon a body are in equilibrium, and a point in it is fixed, the resultant of the forces must pass through this point.

69. In order to give a fuller exposition of some of the foregoing results; let P_1, P_2, &c., be forces acting on any points in a plane rigid surface; A_1 the point at which P_1 is applied; Oy, Ox rectangular axes; Op_1 the perpendicular on the direction of A_1P_1; A_1X_1 and A_1Y_1 the components of A_1P_1 resolved in the directions of the axes. A similar construction will apply to all the other forces.

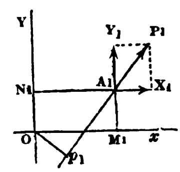


Fig. 35.

Now, as in Art. **63.**, the sum of the forces resolved in the direction of Ox will be equal to the force of the resultant resolved in the same direction, and so on similarly with the resolved forces in the direction Oy, hence we have

$$X=X_1+X_2+X_3+\&c. \ldots (1),$$

$$\text{and } Y=Y_1+Y_2+Y_3+\&c. \ldots (2);$$

but from the right angled triangle $A_1X_1P_1$, we have

$$A_1X_1=A_1P_1 \times \cos\alpha_1, \text{ and } A_1Y_1=A_1P_1 \times \sin\alpha_1, \text{ that is}$$

$$X_1=P_1 \cos\alpha_1, \text{ and } Y_1=P_1 \sin\alpha_1,$$

and similarly $X_2=P_2 \cos\alpha_2$, $Y_2=P_2 \sin\alpha_2$, and so on to the other forces; therefore by substituting in the preceding expression (1) and (2), we obtain (1) and (2) of Art. **63.**; and hence, also, eqs. (3) and (4) of Art. **63.**, hold true. Moreover, since $P_1 \times Op_1$ is the moment of P_1, and so on to the other forces, therefore the general equation of moments (4) Art. **66.**, also holds true.

WORK OF FORCES ACTING IN THE SAME PLANE.

70. To estimate the work of a pressure, when its line of action is not in the direction of the line in which the point of application moves.

Let PA represent the pressure, and let the point of application A move from A to O by the action of this pressure constantly acting parallel to itself. Complete the parallelogram AXPY, and draw Ap perpendicular to OP′ or AP. The pressure AP is equivalent to the two pressures AX and AY, but the latter pressure does no work because no motion takes place in its direction, therefore the whole work is performed by the pressure AX;

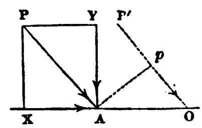


Fig. 36.

$$\therefore \text{ work done from A to O} = \text{pressure} \times \text{space}$$
$$= \text{AX} \times \text{AO}.$$

But from the similar triangles APX and OAp,

$$\text{AX} \times \text{AO} = \text{AP} \times \text{O}p = \text{P} \times \text{O}p,$$
$$\therefore \text{ work done from A to O} = \text{P} \times \text{O}p \ldots (1).$$

Now Op may be regarded as the *virtual* motion of P, or the motion of P estimated in the direction in which it acts, hence if we call Op the space described by P, then P × Op will be the work of P; taking this aspect of the subject the preceding result may be expressed as follows: the work of a pressure in moving the point of its application over a given space, not in the line of its action, is equal to the work of the pressure estimated in the direction in which the pressure acts.

If U = the work done from A to O; $\angle \text{PAX} = \alpha$; $s = \text{O}p$; and $\text{S} = \text{AO}$; then

$$\text{U} = \text{P} \,.\, s \ldots (2);$$

but $s = \text{S} \,.\, \cos\alpha$,

$$\therefore \text{U} = \text{P} \,.\, \text{S} \,.\, \cos\alpha \ldots (3).$$

71. *To estimate the work of any number of pressures applied to a point.*

Let P_1, P_2, &c., be the pressures applied to the point A (see *fig.* 33. Art. **66.**), and let A be moved to O, while the pressures remain constant and parallel to themselves, then we have, by eq. (1), Art. **62.**,

$$\text{X} = \text{P}_1 \cos\alpha_1 + \text{P}_2 \cos\alpha_2 + \&\text{c.}$$

Multiplying by AO, we have

$$\text{X} \,.\, \text{AO} = \text{P}_1 \,.\, \text{AO} \,.\, \cos\alpha_1 + \text{P}_2 \,.\, \text{AO} \,.\, \cos\alpha_2 + \&\text{c.},$$

but AO . $\cos\alpha_1 = Ap_1 = s_1$, AO . $\cos\alpha = Ap_2 = s_2$, and so on; hence we have, putting S for AO,

$$X \,.\, S = P_1 \,.\, s_1 + P_2 \,.\, s_2 + \&c. \ldots (1).$$

Here X . S is the whole work of the pressures estimated in the direction of motion, $P_1 \,.\, s_1$ is the work of P_1 estimated in the direction in which it acts (see eq. (2), Art. **70.**), and so on to the other products. Hence THE WHOLE WORK DONE IS EQUAL TO THE SUM OF THE WORKS OF THE DIFFERENT PRESSURES ESTIMATED IN THEIR RESPECTIVE DIRECTIONS. Or we may more simply express this result by saying, that THE WORK OF THE RESULTANT IS EQUAL TO THE SUM OF THE WORKS OF THE DIFFERENT COMPONENTS.

It is to be observed that each term in eq. (1) is to be taken positively or negatively, according as the direction of the corresponding pressure acts towards the direction of motion or contrary to it.

72. *To estimate the work of any number of pressures applied to any points in a plane rigid surface.*

Let P_1, P_2, &c. be any number of pressures acting at the points A_1, A_2, &c. of a rigid surface, let it be moved over the space AO in the direction AX, while the pressures remain constant and parallel to themselves; then the points of application, A_1, A_2, &c. will be respectively moved over the spaces A_1O_1, A_2O_2, &c., and which are respectively equal and parallel to AO. Hence we have, by eq. (1), Art. **69.**,

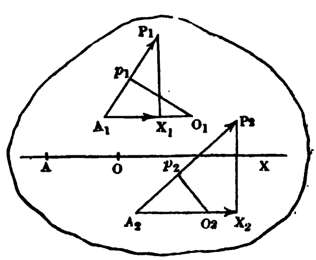


Fig. 37.

$$X = X_1 + X_2 + \&c.$$

multiplying by AO, we have

$$X \,.\, AO = X_1 \,.\, AO + X_2 \,.\, AO + \&c.$$

But from the similar triangles $A_1P_1X_1$ and $A_1O_1p_1$,

$$A_1X_1 \times A_1O_1 = A_1P_1 \times A_1p_1;$$

that is,

$$X_1 \times AO = P_1 \,.\, s_1;$$

in like manner from the similar triangles $A_2P_2X_2$ and $A_2O_2p_2$,

$$X_2 \times AO = P_2 \,.\, s_2, \text{ and so on};$$

$$\therefore\ X \,.\, S = P_1 \,.\, s_1 + P_2 \,.\, s_2 + \&c. \ldots (1),$$

which is the same result as that of eq. (1), Art. **71.**

73. If U represent the whole work or X . S, and U_1 the sum of the works done by the different pressures in the same direction, and U_2 those done in the contrary direction; then eq. (1), Art. **72.** becomes

$$U=U_1-U_2 \ldots (1).$$

If the pressures are in equilibrium, then X=O, and eq. (1), Art. **72.**, becomes

$$P_1 \, . \, s_1+P \, . \, s_2+\&c.=0 \ldots (2).$$
$$\text{or } U_1-U_2=0.$$

74. ***Principle of virtual velocities.***—When the displacement AO or S is very small, s_1, s_2, &c. are called the ***virtual velocities*** of the pressures P_1, P_2, &c., and then eq. (1), Art. **72.**, represents the equation of virtual velocities, where it is now to be observed that the pressures may vary both with respect to direction and intensity, but that they are regarded as constant just for the instant at which the equality holds true.

75. ***To estimate the work of a pressure when the rigid surface turns about a fixed centre.***

Let O be the fixed centre, P_1 the pressure applied to the point A in the direction $p_1 P_1 a_1$; let A be moved through the small arc AA_1, then the resultant pressure R_1 will be in the direction AA_1; from O and A_1 let fall the perpendiculars Op_1 and A_1a_1. Now because AA_1 is very small, it may be regarded as a straight line perpendicular to OA; hence the triangles AA_1a_1 and AOp_1 are similar.

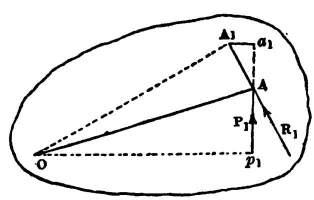


Fig. 38.

As no motion takes place in the direction OA, therefore no work is done in this direction, and therefore the whole of the work is done by R_1 in the direction AA_1;

$$\therefore \text{ Work done from A to } A_1=R_1 \times AA_1;$$

by the equality of moments,

$$R_1 \times OA=P_1 \times Op_1;$$

and by the similar triangles AA_1a_1 and AOp_1,

$$\frac{AA_1}{OA}=\frac{Aa_1}{Op_1};$$

therefore, by multiplying the last two equalities,

$$R_1 \times AA_1=P_1 \times Aa_1;$$

that is,

$$\text{Work done from A to } A_1=P_1 \times Aa_1$$
$$=P_1 \times s_1 \ldots (1);$$

which is the same result as eq. (1), Art. 70., the space s_1 in this case being taken very small, or, what is the same thing, s_1 being the virtual velocity of P_1.

76. The principle demonstrated in the preceding article holds true for any angle of rotation.

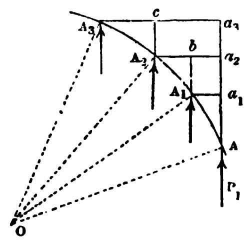


Fig. 39.

As OA revolves let the pressure P_1 act constantly parallel to its first direction; let A, A_1, A_2, &c. be the consecutive positions of the force; draw $A_1 a_1$, $A_2 a_2$, &c. perpendicular to $P_1 A$ produced, the direction of the force P_1; then by eq. (1), Art. 75.,

$$\text{Work done from A to } A_1 = P_1 \times A a_1,$$

$$\text{,, ,, ,, } A_1 \text{ to } A_2 = P_1 \times A_1 b = P_1 \times a_1 a_2,$$

$$\text{,, ,, ,, } A_2 \text{ to } A_3 = P_1 \times A_2 c = P_1 \times a_2 a_3,$$

and so on;

$$\therefore \text{ Work done from A to } A_3 = P_1 \times A a_1 + P_1 \times a_1 a_2 + P_1 \times a_2 a_3$$
$$= P_1 (A a_1 + a_1 a_2 + a_2 a_3)$$
$$= P_1 \times A a_3,$$

and so on to any number of consecutive positions; hence we have generally, *the work done by the parallel pressure* P_1 *in turning* OA *through any angle is equal to the pressure multiplied by the space through which it moves in the line of its direction.*

77. The propositions contained in the two foregoing Articles may be extended to any number of pressures, P_1, P_2, &c.

By eq. (4), Art. 66., we have

$$R \cdot r = P_1 \cdot p_1 + P_2 \cdot p_2 + \&c.$$

Suppose the whole rigid plane, in which the forces act, to be turned through a small angle $AOA_1 = \theta$, see *fig.* 38., then the lines drawn from O to the different points of application of the forces will all move through the same angle; also let $s_1, s_2, \ldots, s$, be the virtual velocities of $P_1, P_2, \ldots, R$ respectively. Now from the similar triangles AOp_1 and AA_1a_1, we have

$$\frac{Op_1}{OA} = \frac{A a_1}{AA_1} = \frac{A a_1}{OA \times \theta};$$

$$\therefore\ Op_1 = \frac{A a_1}{\theta}, \text{ that is, } p_1 = \frac{s_1}{\theta};$$

$$\text{in like manner, } p_2 = \frac{s_2}{\theta},\ p_3 = \frac{s_3}{\theta}, \ldots, \text{ and } r = \frac{s}{-};$$

hence by substitution,

$$R \cdot \frac{s}{\theta} = P_1 \cdot \frac{s_1}{\theta} + P_2 \cdot \frac{s_2}{\theta} + \&c.;$$

$$\therefore R \cdot s = P_1 \cdot s_1 + P_2 \cdot s_2 + \&c. \ldots (1),$$

where each of these different products represents the work of each pressure done through the virtual space over which it moves. By reasoning as in Art. 76., it follows that the preceding formula is not limited with respect to the angle of rotation.

Parallel Pressures.

Although the fundamental proposition relative to parallel pressures has been demonstrated in prop. 1., Art. 51., yet it will be instructive to show how this proposition may be derived from the parallelogram of forces.

78. Let two parallel pressures P and Q act at the points A and B

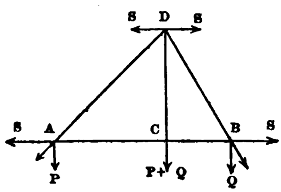


Fig. 40.

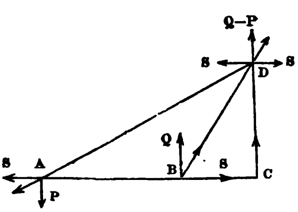


Fig. 41.

respectively of a rigid plane; it is required to find the direction and magnitude of their resultant.

At the points A and B let equal and opposite forces S be applied acting in the line AB; these forces will obviously not affect the system. P and S acting at A will have a resultant in the direction AD, and Q and S acting at B will have a resultant in the direction BD. In the second figure the forces act in contrary directions, and Q is supposed to be greater than P; and therefore in this case the resultant BD must lie nearer to Q than the resultant AD does to P; and hence BD and AD must meet at some point. Suppose these resultant forces to act at D, and let them be resolved into their original components, viz., the resultant in AD into P and S, and the resultant in BD into Q and S; the portions S and S will destroy each other, leaving the resultant P+Q acting in the direction DC in the first figure, and the resultant Q—P acting in the direction CD in the second figure.

Now from the triangle of pressures (see Art. 58.) we have, first from the triangle ACD,

$$P : S :: CD : AC;$$

and second from the triangle BCD,

$$S : Q :: BC : CD;$$

therefore, compounding these proportions,

$$P : Q :: BC : AC;$$

$$\therefore P \times AC = Q \times BC \ldots (1).$$

When AB is perpendicular to the direction of the forces, AC and BC become the *arms* of the forces, and then this equality expresses the equation of moments.

79. If C be a fixed point or fulcrum, its resistance will destroy the resultant, in DC, passing through it, and P, Q will be in equilibrium about this point when their moments are equal to each other. This is *the principle of the lever.*

80. Let P_1 and P_2 be two parallel pressures, acting at P_1 and P_2 in the line Ox, and R_1 their resultant; then by eq. (1),

$$P_1 \times P_1R_1 = P_2 \times P_2R_1,$$

$$\therefore P_1 \times (OR_1 - OP_1) = P_2 \times (OP_2 - OR_1),$$

$$\therefore (P_1 + P_2) \times OR_1 = P_1 \times OP_1 + P_2 \times OP_2,$$

$$\therefore R \times OR_1 = P_1 \times OP_1 + P_2 \times OP_2 \ldots (2).$$

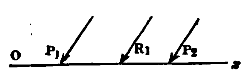


Fig. 42.

81. To find the resultant of any number of parallel pressures, P_1, P_2, &c., acting at any point in a plane.

Take Ox any line in the plane, and produce the pressures until they intersect this line, as shown in the figure; now as these forces may act at any point of their direction, we may suppose P_1 to be applied at p_1, P_2 at p_2, and so on. Let R_1 be the resultant of P_1 and P_2, R_2 the resultant of R_1 and P_3, and so on. Then by Art. 51. or 78. we have

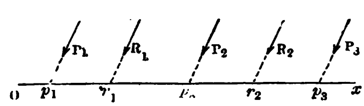


Fig. 43.

$$R_1 = P_1 + P_2,$$

$$R_2 = R_1 + P_3 = P_1 + P_2 + P_3, \text{ and so on.}$$

Put O$p_1 = p_1$, O$p_2 = p_2$, and so on; O$r_1 = r_1$, O$r_2 = r_2$, and so on; then by eq. (2), Art. 80.,

$$R_1 \cdot r_1 = P_1 \cdot p_1 + P_2 \cdot p_2.$$

In like manner

$$R_2 \cdot r_2 = R_1 \cdot r_1 + P_3 \cdot p_3$$
$$= P_1 \cdot p_1 + P_2 \cdot p_2 + P_3 \cdot p_3;$$

and so on to any number of pressures. Hence we have generally, putting R for the resultant of the whole pressures, and r for its distance from O,

$$R \cdot r = P_1 \cdot p_1 + P_2 \cdot p_2 + \&c. \ldots (1);$$

but $R = P_1 + P_2 + \&c.$,

$$\therefore (P_1 + P_2 + \&c.)\, r = P_1 \cdot p_1 + P_2 \cdot p_2 + \&c. \ldots (2);$$

$$\therefore r = \frac{P_1 p_1 + P_2 p_2 + \&c.}{P_1 + P_2 + \&c.} \ldots (3),$$

which gives the distance at which the resultant acts from O. When some of the pressures in the foregoing formulæ act in opposite directions they must be taken negative; and also when some of the forces act on the left side of O, their moments must be taken negative.

82. To find the resultant of any number of parallel pressures P_1, P_2, &c. acting at any points in space.

Let P_1, P_2, P_3, &c. be the points of application; let Ox and Oy be the rectangular axes in any assumed plane xOy, and Oz an axis perpendicular to this plane; join P_1 and P_2, and produce the line until it cuts the plane xOy in C; let R_1 be the resultant of P_1 and P_2; join R_1 and P_3, and let R_2 be the resultant of R_1 and P_3; and so on; from P_1, R_1, P_2 let fall the perpendiculars $P_1 p_1$, $R_1 r_1$, $P_2 p_2$ on the plane xOy; then by eq. (2), Art. **80.**, we have

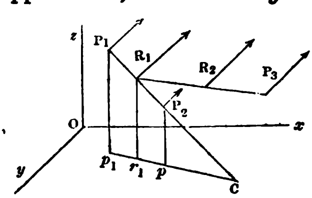


Fig. 44.

$$R_1 \times CR_1 = P_1 \times CP_1 + P_2 \times CP_2,$$

$$\therefore R_1 = P_1 \times \frac{CP_1}{CR_1} + P_2 \times \frac{CP_2}{CR_1}.$$

Now by the similar triangles, CP_1p_1, CR_1r_1, and CP_2p_2,

$$\frac{CP_1}{CR_1} = \frac{P_1 p_1}{R_1 r_1}, \quad \frac{CP_2}{CR_1} = \frac{P_2 p_2}{R_1 r_1};$$

therefore by substitution,

$$R_1 = P_1 \times \frac{P_1 p_1}{R_1 r_1} + P_2 \times \frac{P_2 p_2}{R_1 r_1};$$

$$\therefore R_1 \times R_1 r_1 = P_1 \times P_1 p_1 + P_2 \times P_2 p_2.$$

Or putting Z_1 for $R_1 r_1$, z_1 for $P_1 p_1$, and z_2 for $P_2 p_2$,

$$R_1 \cdot Z_1 = P_1 \cdot z_1 + P_2 \cdot z_2.$$

In like manner we have for the resultant R_2 of the pressures R_1 and P_3,

$$R_2 . Z_2 = R_1 . Z_1 + P_3 . z_3$$
$$= P_1 . z_1 + P_2 . z_2 + P_3 . z_3,$$

and so on to the resultant R_3 and P_4, &c., so that, putting $R . Z$ for for the moment of the last resultant, we have generally

$$R . Z = P_1 . z_1 + P_2 . z_2 + \&c. \dots (1),$$

where the pressures which tend to move in a contrary direction must be taken negative.

In like manner we have for the planes zox and zoy respectively

$$R . Y = P_1 . y_1 + P_2 . y_2 + \&c. \dots (2),$$

$$\text{and } R . X = P_1 . x_1 + P_2 . x_2 + \&c. \dots (3).$$

Now as in Art. 81., we have

$$R = P_1 + P_2 + \&c.,$$

therefore from eq. (1)

$$Z = \frac{P_1 . z_1 + P_2 . z_2 + \&c.}{P_1 + P_2 + \&c.} \dots (4),$$

which determines the distance of the point at which the resultant acts from the plane xoy; in like manner Y is found from eq. (2) and X from eq. (3), which give the distances of the point at which the resultant acts, measured in reference to the planes xoz and yoz respectively.

83. The point at which the resultant acts is called the *centre of parallel pressures.* The product of a pressure by its distance from a plane is called the moment of that pressure in relation to that plane; according to this definition, therefore, eq. (1) shows that *the sum of the moments of the different parallel pressures, considered in relation to a given plane, is equal to the moment of the resultant.*

84. Now the value of Z given by eq. (4) is independent of the *directions* of the parallel pressure. Hence it follows that the position of the centre of pressure will remain unchanged whatever may be the direction of the pressures, provided that their magnitudes and points of application remain the same.

Work of parallel Pressures.

85. Let the pressures P_1, P_2, P_3, &c. act perpendicular to the plane xoy (see *fig.* 44.), suppose P_1 to be moved through the space

h_1, P_2 through h_2, and so on, and R through h; then by eq. (1), Art. **82.**,

$$R \cdot Z = P_1 \cdot z_1 + P_2 \cdot z_2 + \&c.$$

But we have for the new position of the forces, R, P_1, P_2, &c. respectively $Z+h$, z_1+h_1, z_2+h_2, &c.

$$\therefore\ R(Z+h) = P_1(z_1+h_1) + P_2(z_2+h_2) + \&c.;$$

subtracting the first equality from the second,

$$R \cdot h = P_1 \cdot h_1 + P_2 \cdot h_2 + \&c.,$$

$$\therefore\ (P_1+P_2+\&c.)h = P_1 \cdot h_1 + P_2 \cdot h_2 + \&c. \ldots (1).$$

but the different products in this expression are respectively equal to the work of their corresponding pressures; hence *the work done by the sum of the parallel pressures is equal to the sum of the works severally done by the pressures;* observing that the works done in a contrary direction must be taken negatively.

86. If U be put for the work of the resultant, U_1 for the sum of the works of the pressures done positively, and U_2 for those which are done negatively; then eq. (1) becomes

$$U = U_1 - U_2 \ldots (1).$$

87. If the pressures P_1, P_2, &c. are in equilibrium, then $U=0$, and

$$U_1 - U_2 = 0, \text{ or } U_1 = U_2 \ldots (1).$$

LIMITING ANGLE OF FRICTION.

88. Let A be a material particle (see *fig.* 36.) moved on a rough plane AO by the action of a pressure P represented by AP, then by Art. **70.**, AX represents the effective force tending to give motion to the particle, and AY the effective pressure on the plane. Let f=the coefficient of friction (see Art. **23.**), and $\angle PAY = \theta$, then we have

$$\text{Pressure on the plane} = AY = AP \cdot \cos\theta = P\cos\theta;$$

$$\therefore\ \text{Resistance of friction} = fP\cos\theta;$$

Effective pressure tending to move the particle in opposition to the resistance of friction $=AX=P\sin\theta$. Now motion will or will not take place, according as this pressure is greater or less than the resistance of friction; and when motion is upon the point of taking place, the one must be equal to the other, and then the

angle θ is called the limiting angle of resistance; in this case therefore, we have

$$f\text{P} \cos \theta = \text{P} \sin \theta,$$

$$\therefore f = \frac{\sin \theta}{\cos \theta} = \tan \theta.$$

The magnitude of this angle has been determined by experiment for different rubbing surfaces (See "Moseley's Engineering," p. 149.).

This result shows that the coefficient of friction is equal to the tangent of the limiting angle of resistance. If the pressure P be applied within this angle, then no motion can take place, however great that pressure may be; and on the contrary, if the pressure be applied without this angle, then motion will take place, however small that pressure may be.

CHAP. V.

CENTRE OF GRAVITY.

89. EVERY heavy body is composed of an indefinite number of particles, each of which is acted upon by the force of gravity in a direction perpendicular to the horizon. The sum of all these parallel forces is evidently the weight of the body. Now there must be a point, where a single force, equal to the weight of the body, being applied, will produce the same effect, as the force of gravity acting upon the various particles composing the body;— this point is called the centre of gravity of the body.

If the pressures P_1, P_2, &c. in Art. **82.**, be produced by a system of heavy particles whose weights are respectively represented by the units in these pressures, then the centre of gravity of the system will correspond with *the centre of parallel pressures* there determined, and the equations (1), (2), (3), (4), &c. will hold true in relation to the position of the centre of gravity of any system of bodies or parts of the same body. Hence it follows that a body or a system of bodies can only have one point as the centre of gravity, which in general lies in the intersection of three planes perpendicular to one another. Since the centre of gravity of a body or a system of bodies lies in the direction of the resultant of the parallel pressures of the parts composing the system, it follows,—1. That if the centre of gravity be supported the

system will remain at rest, and *vice versâ*. 2. That if any body, acted upon by the force of gravity, tend to turn a lever, we may regard the weight of the whole body to be collected in its centre of gravity. 3. That the centre of gravity will always descend to the lowest point possible. 4. That the centre of gravity of all regular, or symmetrical bodies is in their centre of magnitude.

90. To find the centre of gravity, G, of two weights, P_1 and P_2, connected by a rigid rod supposed to be without weight.

Fig. 45.

If the point G be supported, the bodies will be in equilibrium; then we have by the principle of the equality of moments, expressed by eq. (1), Art. **78.**,

$$P_1 \times P_1G = P_2 \times P_2G.$$

Let $P_1 P_2 = l$, the distance between the weights; and $x = P_1G$, the distance of the centre of gravity from the weight P_1; then $P_2G = l - x$, and the preceding equality becomes

$$P_1 \times x = P_2 \times (l - x);$$

$$\therefore\ x = \frac{P_2 \times l}{P_1 + P_2}.$$

Or thus,

Taking any point O, in the direction of the rod, as an axis of moments; put $OP_1 = p_1$, $OP_2 = p_2$, and $OG = x$. Here by eq. (2) Art. **80.**, the moment of the sum of the weights acting in their centre of gravity, G, is equal to the sum of the moments of the weights taken separately,

Fig. 46.

$$\therefore\ (P_1 + P_2)x = P_1 \cdot p_1 + P_2 \cdot p_2;$$

$$\therefore\ x = \frac{P_1 \cdot p_1 + P_2 \cdot p_2}{P_1 + P_2}.$$

Now when O is taken at P_1, we then have, $OP_1 = 0$, and in this case the expression becomes

$$x = \frac{P_2 \cdot p_2}{P_1 + P_2} = \frac{P_2 \times l}{P_1 + P_2},$$

which is the same result as before determined.

91. If m_1 be put for the volume of the weight P_1, m_2 for that of P_2, and so on, and m for the volume of the whole body making up the weights P_1, P_2, &c., then when the material composing these bodies

is the same, their weights will be proportional to their volumes, and consequently in determining the centre of gravity of a body, or of a system of bodies, composed of the same material, we may use the volumes of the bodies in the place of their weights.

92. To find the centre of gravity of any number of bodies, considered as points, in the same plane.

Let W_1, W_2, W_3, &c., be the weights of the bodies; take any two straight lines Ax, and Ay at right angles to each as axes of co-ordinates. Join W_1 W_2, and let G_1 be the centre of gravity of W_1 and W_2; join G_1 W_3, and let G_2 be the centre of gravity of the three weights W_1, W_2, W_3; and so on. From W_1, G_1, W_2, G_2, &c., let fall $W_1 a$, $G_1 b$, $W_2 c$, $G_2 d$, &c., perpendicular to Ax. Let y_1, y_2, y_3, &c., be the distances of

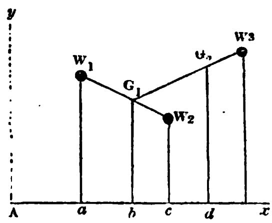


Fig. 47.

W_1, W_2, W_3, &c., from Ax, and x_1, x_2, x_3, &c., their distances from Ay; also let y and x be the distances of the centre of gravity of the system from Ax and Ay respectively. Now if we find expression for y and x, the centre of gravity of the system will be determined.

By the equality of moments,

$$W_1 \times W_1 G_1 = W_2 \times W_2 G_1,$$

$$\therefore \frac{W_1}{W_2} = \frac{W_2 G_1}{W_1 G_1} = \frac{cb}{ab}, \text{ by similar figures,}$$

$$= \frac{x_2 - Ab}{Ab - x_1};$$

solving for Ab, we get

$$Ab = \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2}.$$

Now in finding the centre of gravity of the three bodies W_1, W_2, W_3, we may suppose the two former to act in their centre of gravity G_1; hence we have

$$Ad = \frac{(W_1 + W_2)Ab + W_3 x_3}{W_1 + W_2 + W_3}, \text{ substituting for } Ab,$$

$$= \frac{W_1 x_1 + W_2 x_2 + W_3 x_3}{W_1 + W_2 + W_3};$$

and so on for any number of bodies; hence we have generally,

$$x = \frac{W_1 x_1 + W_2 x_2 + \ldots + W_n x_n}{W_1 + W_2 + \ldots + W_n} \ldots (1),$$

which gives the distance of the centre of gravity of the system from Ay. If the system be conceived to turn round on Ay as an axis of motion, the numerator of the right hand member of this expression will be the sum of the moments of the weights, and then the whole formula becomes an obvious application of the simple principle of the equality of moments.

In precisely the same way, we find

$$y = \frac{w_1 y_1 + w_2 y_2 + \ldots + w_n y_n}{w_1 + w_2 + \ldots + w_n} \ldots (2)$$

which gives the distance of the centre of gravity of the system from Ax. In this case we may conceive the system to turn round on Ax as an axis of motion.

When the bodies are not in the same plane; if we conceive three planes perpendicular to each other to be drawn on which we may conceive the weights to be projected perpendicularly, the three edges formed by these planes, will become three axes of motion about which the moments of the weights are to be taken, so that, in this case, we derive three equations of precisely the same form as (1) and (2). See eqs. (1), (2), and (3), Art. **82**.

93. To find the centre of gravity of a triangle.

Let ABC be the triangle; bisect AB in D, and join CD. Now CD will bisect all lines, such as *be*, drawn parallel to AB.

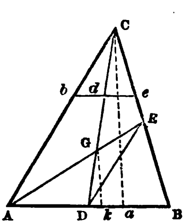


Fig. 48.

The triangle may be conceived to be made up of a series of lines, such as *be*, drawn parallel to AB, having their middle points or centres of gravity lying in CD. Hence the triangle would be balanced upon an edge lying along CD; that is, the centre of gravity of the triangle lies somewhere in this line. In like manner the centre of gravity of the triangle must also lie somewhere in the line AE bisecting the side BC; but it has been shown, that it also lies in CD, therefore it must be in the point G, where these two lines intersect.

Join DE, then because CE is equal to EB and AD to DB, the line DE is parallel to AC, and the triangles DGE and AGC are similar; hence

$$\frac{CG}{GD} = \frac{AC}{DE},$$

but AC is double DE;

$$\therefore \; CG = 2GD;$$

$$\therefore \; GD = \tfrac{1}{3}CD.$$

Thus the distance, GD, of the centre of gravity from the middle of the base is one-third the whole line CD.

Further, if Gk and Ca be drawn perpendicular to the base, the triangles DGk, DCa will be similar; and since DG is one-third DC, therefore Gk will be one-third Ca. Hence *the distance of the centre of gravity of a triangle from the base is one-third the whole elevation.*

94. To find the centre of gravity of a pyramid having a triangular base.

Let ABCD be the triangular pyramid. Let ADE be a plane passing through the edge AD and bisecting CB; a the centre of gravity of the face ABC, and r that of BCD; join Da and Ar, intersecting each other in G; then G will be the centre of gravity of the pyramid.

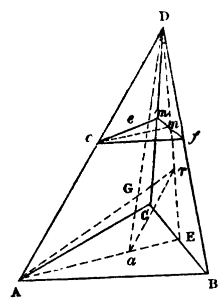


Fig. 49.

We may conceive the pyramid to be made up of thin plates, such as cfm, parallel to ABC. Now because mf is parallel to CB, and CE is equal to EB, therefore mn is equal to nf; let Da meet cn in e, then because cn is parallel to AE, the line Dea will divide these two lines proportionally, that is, since Aa=2aE, so therefore $ce=2en$; hence e will be the centre of gravity of the triangle cfm. In like manner it may be shown that the centre of gravity of every section parallel to ABC is in the line Da; therefore the centre of gravity of the whole pyramid must lie in the line Da; and for the same reason it must also lie in the line Ar; therefore it must be in the point G, where these two lines intersect.

Since Ea=$\frac{1}{3}$EA, and Er=$\frac{1}{3}$ED, Art. **93**., therefore ar is paralle to AD, and the triangles arE, ADE are similar,

$$\therefore\ \frac{\mathrm{AD}}{ar}=\frac{\mathrm{AE}}{a\mathrm{E}};$$

but AE is three times aE;

$$\therefore\ \mathrm{AD} \text{ is three times } ar.$$

Also the triangles AGD, aGr are similar, and

$$\therefore\ \frac{\mathrm{GD}}{\mathrm{G}a}=\frac{\mathrm{AD}}{ar},$$

but AD is three times ar;

$$\therefore\ \mathrm{GD}=3\mathrm{G}a;$$

$$\therefore\ \mathrm{G}a=\tfrac{1}{4}\mathrm{D}a.$$

Thus the distance, Ga, of the centre of gravity from the base is one-fourth the whole line Da.

Further, if perpendiculars be drawn from G and D upon the base ABG, these perpendiculars will have the same ratio to each other, as the lines aG and aD; hence it follows, that *the distance of the centre of gravity of a pyramid from the base is one-fourth the whole elevation.*

95. The above rule holds good whatever may be the form of the base of the pyramid or the conical body.

As every conical body may be conceived to be made up of an infinite number of three-sided pyramids of the same height, and having their centres of gravity at the same distance from the base, that is, at one-fourth the whole perpendicular height, it therefore follows, that the centre of gravity of the whole conical body must also lie at this distance from the base.

96. To find the centre of gravity of a frustum ABED of a cone.

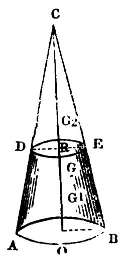


Fig. 50.

Let ACB be the complete cone, G its centre of gravity; G_2 the centre of gravity of the top cone CDE; and G_1 that of the frustum; then taking O as the centre of moments, we have, Art. **91.**,

$$\text{CG} \times \text{cone ABO} = \text{CG}_2 \times \text{cone DEC} + \text{CG}_1 \times \text{frust. ABED},$$

$$\therefore\ \text{CG}_1 = \frac{\text{CG} \times \text{cone ABC} - \text{CG}_2 \times \text{cone DEC}}{\text{frustum ABED}}.$$

Put CO$=a$, CR$=a_1$, and $m=$cone ABC; then CG$=\frac{3}{4}a$, CG$_2$ $=\frac{3}{4}a_1$; and as similar solids are to each other as the cubes of their like dimensions, cone DEC$=\frac{a_1^3}{a^3}m$, frustum ABED$=m-\frac{a_1^3}{a^3}m$ $=m\left(1-\frac{a_1^3}{a^3}\right)$; therefore by substitution

$$\text{CG}_1 = \frac{\frac{3}{4}a \times m - \frac{3}{4}a_1 \times \frac{a_1^3}{a^3}m}{m\left(1-\frac{a_1^3}{a^3}\right)} = \frac{3}{4} \cdot \frac{a^4 - a_1^4}{a^3 - a_1^3},$$

which is the expression required.

This formula will obviously hold good for any truncated pyramid, whatever may be the form of its base.

97. To find the height of the centre of gravity of any frustum of a pyramid.

Let k^2=the area of the base AB (see *fig.* 50.); k_1^2=the area of the top DE; h=the height of the frustum; x=CR the height of the top pyramid; then (Tate's "Geo. and Mensuration," p. 75.) we have

$$k^2 : k_1^2 :: (h+x)^2 : x^2,$$

$$\therefore\ x=\frac{hk_1}{k-k_1},\ \text{and}\ h+x=\frac{hk}{k-k_1}.$$

Now taking the moments with reference to the base AB, we have

$$\text{Moment whole pyramid}=\frac{1}{3}k^2(h+x)\times\frac{1}{4}(h+x)$$

$$=\frac{1}{12}\cdot\frac{h^2k^4}{(k-k_1)^2}.$$

$$\text{Moment top pyramid}=\frac{1}{3}k_1^2x\times\left(\frac{1}{4}x+h\right)$$

$$=\frac{1}{12}\cdot\frac{h^2k_1^3(4k-3k_1)}{(k-k_1)^2};$$

$$\therefore\ \text{Moment frustum}=\text{Moment whole pyr.}-\text{Moment top pyr.}$$

$$=\frac{h^2}{12}\ \frac{k^4-4kk_1^3+3k_1^4}{(k-k_1)^2}$$

$$=\frac{h^2}{12}(k^2+2kk_1+3k_1^2)\ldots(1).$$

Let y=the height of the centre of gravity of the frustum; then

$$\text{Moment frustum}=y\times\text{solidity frustum}$$

$$=y\times\frac{1}{3}\{k^2(h+x)-k_1^2x\}$$

$$=y\times\frac{h}{3}(k^2+kk_1+k_1^2);$$

hence we have, by equating this expression with eq. (1),

$$y=\frac{h}{4}\cdot\frac{k^2+2kk_1+3k_1^2}{k^2+kk_1+k_1^2}\ldots(2).$$

This formula also holds true when k_1 is greater than k, which may be readily verified by subtracting y from h.

98. When the solid is a frustum of a cone, the radii of whose ends are r and r_1, then $k^2=\pi r^2$, and $k_1^2=\pi r_1^2$; in this case, there-

fore, the above expression holds good by merely substituting r for k and r_1 for k_1.

99. To find the centre of gravity of a trapezoid ABCD.

Let DC and AB be the parallel sides; bisect DC in M, and AB in L; join ML; then the centre of gravity of the trapezoid must lie in this line. From D draw DK parallel to CB. Let G_1 be the centre of gravity of the parallelogram DCBK, and G_2 that of the triangle ADK; join $G_1 G_2$; cutting ML in G; then G will be the centre of gravity of the trapezoid.

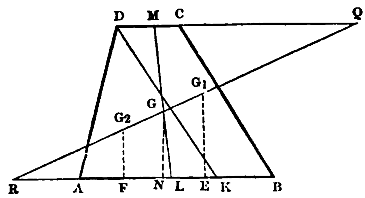


Fig. 51.

Let fall G_2F, GN, and G_1E, perpendiculars to AB; then taking AB as the axis of moments, we have

$$GN \times \text{area trap. } ABCD = G_2F \times \text{area } ADK + G_1E \times \text{area } DCBK.$$

Now put h=the perpendicular distance between AB and DC; AB$=b$; DC$=b_1$; then, Art. **93.**, $G_2F=\frac{1}{3}h$; $G_1E=\frac{1}{2}h$; area ADK $=AK \times \frac{1}{2}h=(b-b_1)\times\frac{1}{2}h$; area DCBK$=b_1 h$; area ABCD$=(b+b_1)\times\frac{1}{2}h$; therefore, by substituting these values in the above equality, we have

$$GN \times (b+b_1)\frac{h}{2}=\frac{h}{3}\times(b-b_1)\frac{h}{2}+\frac{h}{2}\times b_1 h,$$

$$\therefore\ GN=\frac{h(b+2b_1)}{3(b+b_1)};$$

which gives the distance of the centre of gravity of the trapezoid, from the side AB.

100. To find the centre of gravity of a trapezium by geometrical construction.

On the sides DC and AB produced, take CQ=AB, and AR=DC; join RQ, cutting ML in G; then G will be the centre of gravity required; which may be readily proved from the expression for GN just determined.

101. The formula here found also gives the height of the centre of gravity of an embankment, having its end sections parallel to each other, as shown in the annexed figure.

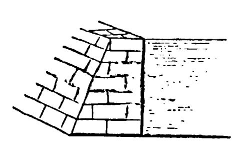


Fig. 52.

102. To find the centre of gravity of a wedge ABCDE.

Let DGS be a plane section parallel to the end CBP; then DBG

will be a prism, whose centre of gravity is at one-third the perpendicular height; and ASD will be a pyramid, whose centre of gravity is at one-fourth the perpendicular height.

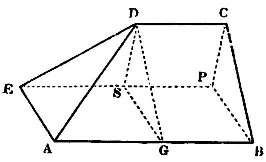


Fig. 53.

Put L=the length AB, l=the length of the edge DC, B=the breadth of the base AE, h=the perpendicular height of the wedge; and y=the height of its centre of gravity; then the solidity of the wedge$=\frac{1}{6}\text{B}h(l+2\text{L})$. See "Tate's Geo. and Mensuration," p. 170.

Momt. wedge=Momt. pyr. ASD+Momt. prism DBG

$$\text{But, Momt. wedge}=y\times\tfrac{1}{6}\text{B}h(l+2\text{L});$$

$$\text{Momt. pyr. ASD}=\tfrac{1}{4}h\times\tfrac{1}{3}h\text{B}(\text{L}-l)$$
$$=\tfrac{1}{12}h^2\text{B}(\text{L}-l);$$

$$\text{Momt. prism DBG}=\tfrac{1}{3}h\times\tfrac{1}{2}hl\text{B},$$
$$=\tfrac{1}{6}h^2l\text{B};$$

$$\therefore\ y\times\tfrac{1}{6}\text{B}h(l+2\text{L})=\tfrac{1}{12}h^2\text{B}(\text{L}-l)+\tfrac{1}{6}h^2l\text{B}$$

$$\therefore\ y=\frac{h(\text{L}+l)}{2(l+2\text{L})}.$$

This formula holds good when the edge is longer than the base.

The distance of the centre of gravity from the edge DC will be

$$h-\frac{h(\text{L}+l)}{2(l+2\text{L})}=\frac{h(3\text{L}+l)}{2(l+2\text{L})}.$$

103. To find the centre of gravity of a pontoon or rectangular prismoid ABCDE, whose parallel faces AP and DH, are dissimilar rectangles.

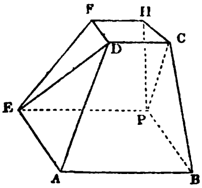


Fig. 54.

Here the solid will be divided into two wedges by a section EDCP passing through the opposite edges DC and EP.

Put L=AB, l=DC, B=AE, b=DF, h=the perpendicular height of the pontoon, y=the height of its centre of gravity.

$$\text{Momt. pontoon}=y\times\text{solidity},$$
$$=y\times\tfrac{1}{6}h\{\text{LB}+lb+(\text{L}+l)(\text{B}+b)\},^*$$

* See Author's "Geometry and Mensuration," p. 171.

and by Art. **102.**,

$$\text{Momt. wedge ABCE}=\frac{h(\text{L}+l)}{2(l+2\text{L})}\times\tfrac{1}{6}\text{B}h(l+2\text{L}),$$

$$=\frac{h^2}{12}\cdot\text{B}(\text{L}+l),$$

$$\text{Momt. wedge EDHP}=\frac{h(3l+\text{L})}{2(\text{L}+2l)}\times\tfrac{1}{6}bh(\text{L}+2l)$$

$$=\frac{h^2}{12}\cdot b(3l+\text{L}).$$

Hence we have, by making the moment of the whole pontoon equal to the sum of the moments of the two wedges, and reducing

$$y=\frac{h}{2}\cdot\frac{\text{B}(\text{L}+l)+b(3l+\text{L})}{\text{LB}+lb+(\text{L}+l)(\text{B}+b)}.$$

This formula holds good whatever may be the relative dimensions of the parallel faces, which may be verified after the manner explained in Art. **97.**

104. To find the centre of gravity of any curved space ABRN, bounded at the extremities by two vertical straight lines AB and NR.

Draw the horizontal line AJ, intersecting NR produced in the point J; divide AJ into n equal parts, AK, KL, . . ., TJ; draw the vertical lines KD, LF, . . ., TS, cutting off the trapezoids ABDC, CDFE, . . ., YSRN. Put a_0=AB, a_1=CD, a_2=EF, . . ., NR$=a_n$, a=AK=KL=&c., and let $g_1, g_2, \ldots, g_n$, be put for the respective distances of the centres of gravity of the 1st, 2nd, . . ., nth trapezoids from A; also, let g be the distance of the centre of gravity G of the whole surface ABRN from A; then, we have, by the equality of moments,

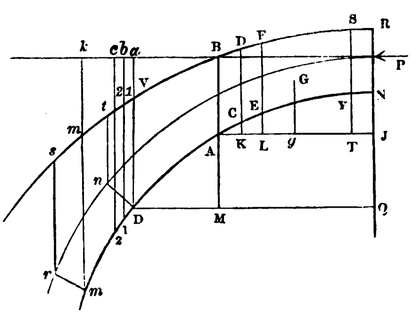


Fig. 55.

$$g=\frac{g_1\times\text{ABDC}+g_2\times\text{CDEF}+\ldots+g_n\times\text{RSPN}}{\text{ABRN}}.$$

But by Art. **99.**, $g_1=\dfrac{a}{3}\cdot\dfrac{a_0+2a_1}{a_0+a_1}$; $g_2=a+\dfrac{a}{3}\cdot\dfrac{a_1+2a_2}{a_1+a_2}$

$$=\frac{a}{3}\cdot\frac{4a_1+5a_2}{a_1+a_2};\text{ and so on; } g_n=(n-1)a+\frac{a}{3}\cdot\frac{a_{n-1}+2a_n}{a_{n-1}+a_n}$$

$$=\frac{a}{3}\cdot\frac{(3n-2)a_{n-1}+(3n-1)a_n}{a_{n-1}+a_n}.$$

Again, ABDC$=(a_0+a_1)\frac{a}{2}$; CDEF$=(a_1+a_2)\frac{a}{2}$; and so on;

RSYN$=(a_{n-1}+a_n)\frac{a}{2}$; and,

$$\text{ABRN}=\frac{a}{2}\{a_0+a_n+2(a_1+a_2+\ldots+a_{n-1})\}.^*$$

Hence we have, by substituting these values in the above expression for g,

$$g=\frac{a}{3}\cdot\frac{(a_0+2a_1)+(4a_1+5a_2)+\ldots+\{(3n-2)a_{n-1}+(3n-1)a_n\}}{a_0+a_n+2(a_1+a_2+\ldots+a_{n-1})}$$

$$=\frac{a}{3}\cdot\frac{a_0+(3n-1)a_n+6\{a_1+2a_2+\ldots+(n-1)a_{n-1}\}}{a_0+a_n+2(a_1+a_2+\ldots+a_{n-1})}\ldots(1).$$

The Author first gave this theorem in his "Exercises on Mechanics."

Example. If $n=4$, then,

$$g=\frac{a}{3}\cdot\frac{a_0+11a_4+6(a_1+2a_2+3a_3)}{a_0+a_4+2(a_1+a_2+a_3)}.$$

105. Let M_n be put for the moment of the surface ABPN about A; then we get from eq. (1),

$$M_n=\text{area ABPN}\times g$$

$$=\frac{a^2}{6}[a_0+(3n-1)a_n+6(a_1+2a_2+\ldots+\overline{n-1}\,a_{n-1})]\ldots(2).$$

This formula may also be established by the following method of proof.

Let $A_1, A_2, \ldots, A_n$, be put for the areas of the spaces ABDC, ABFE, ..., ABRN respectively, and $M, M_2, \ldots, M^n$ for their respective moments about A; then,

$$M_1=A_1\times g_1=\frac{a}{2}(a_0+a_1)\times\frac{a}{3}\cdot\frac{a_0+2a_1}{a_0+a_1}$$

$$=\frac{a^2}{6}(a_0+2a_1);$$

* See the Author's "Geometry and Mensuration," p. 160.

$$\text{M}_2 = \text{A}_1 \times g_1 + \text{A}_2 \times g_2$$
$$= \frac{a^2}{6}(a_0 + 2a_1) + \frac{a}{2}(a_1 + a_2)\left(a + \frac{a}{3} \cdot \frac{a_1 + 2a_2}{a_1 + a_2}\right)$$
$$= \frac{a^2}{6}(a_0 + 5a_2 + 6a_1).$$

Assume the law for the value of M_n, exhibited in eq. (2) to be true, then we shall show that it is also true for M_{n+1},

$$\text{M}_{n+1} = \text{M}_n + \text{A}_{n+1} \times g_{n+1}$$
$$= \text{M}_n + \frac{a}{2}(a_n + a_{n+1})\left(na + \frac{a}{3} \cdot \frac{a_n + 2a_{n+1}}{a_n + a_{n+1}}\right)$$
$$= \text{M}_n + \frac{a^2}{6}\{(3n+1)a_n + (3n+2)a_{n+1}\}$$
$$= \frac{a^2}{6}[a_0 + (3n+2)a_{n+1} + 6(a_1 + 2a_2 + \ldots + na_n)];$$

hence the law assumed for M_n is also true, for M_{n+1}; but we have shown the law to be true for M_1 and M_2; therefore it is true for M_3, and being true for M_3 it is also true for M_4, and so on generally.

106. It will be observed, that in arched structures, the ordinates AB, CD, EF, &c., towards the spring, vary much more in their length than the ordinates, YS, &c., which lie towards the crown; it is desirable, therefore, in order to secure a greater degree of accuracy in the calculation, that the equidistant spaces towards the spring should be taken less than they are taken towards the crown.

Let DVRN be a portion of an arch. Divide DQ into three equal parts, of which DM is one; and let DM and MQ be each divided into n equal parts, and proceed as in the foregoing construction. Put b_0, b_1, . . ., b_n, for the ordinates of the portion DB, A_1 its area, M_1 its moment about D; a_0, a_1, . . ., a_n, the ordinates of the portion AR, A its area, M its moment about A; A^1 the area of the whole portion DR, M^1 its moment about D; a the distance between the ordinates on the portion AR; then, by the principle of moments,

$$\text{M}^1 = \text{M}_1 + \text{A} \times (\text{DM} + \text{A}g)$$
$$= \text{M}_1 + \text{M} + \text{A} \cdot \text{DM} \ldots (3).$$

Now M is given in eq. (2); and similarly,

$$\text{M}_1 = \frac{a^2}{24}\{b_0 + (3n-1)b_n + 6(b_1 + \ldots + \overline{n-1}\,b_{n-1})\},$$

and from eq. (3), by substitution and reduction, we get

$$\text{M}^1 = \frac{a^2}{24}[b_0 + 4a_0 + (3n-1)(b_n + 4a_n) + 6\{(b_1 + 4a_1) + 2(b_2 + 4a_2) +$$
$$\ldots + (n-1)(b_{n-1} + 4a_{n-1})\}] + \text{A} \times \tfrac{1}{2}na \ldots (4),$$

which gives the moment of the surface DVPN about D, whence the distance of its centre of gravity from D is readily found by dividing by $A+A_1$ or A^1.

107. Let Pnr be the extrados of an arch; Dn and mr joints of the arch stones; nt and rs verticals. Put $s=nt$, $p=$the distance between the verticals DV and nt; $q=$the moment of the area DVtn about D; Q=the moment of DNRtn about D; then,

$$Q=M^1-q,$$
$$=M^1-\frac{p^2}{6}(b_0+2s)\ldots(5).$$

Now, area DntRN$=A+A_1+\frac{p}{2}(b_0+s)$,

∴ Distance centre gravity of the mass DntRN from D,

$$=\frac{Q}{\text{area } DntRN}=\frac{M^1-\frac{p^2}{6}(b_0+2s)}{A+A_1+\frac{p}{2}(b_0+s)}\ldots(6).$$

These formulæ are useful in finding the point of rupture in an arch.

108. Given the centre of gravity of a circular arc, to find, (1.) The centre of gravity of a sector of a circle; (2.) That of a circular ring; (3.) That of a segment of a circle.

If DE be the arc of a circle, G its centre of gravity, and C the centre of the circle; then,

$$CG=\frac{\text{radius } CD\times\text{chord } DE}{\text{arc } DE}\ldots(1).$$

(1.) Let it be required to find the centre of gravity of the sector ABC.

Conceive the sector to be divided into an infinite number of triangles, by lines drawn from the centre C to the arc; then the centre of gravity of each triangle will be at a distance from C equal to $\frac{2}{3}$ the radius of the arc AB, and therefore the centres of gravity of all the triangles will lie in a circular arc DE, whose radius CD is equal to $\frac{2}{3}$ CA; now as we may suppose, the weights of all the triangles to be collected in this arc DE, and so uniformly distributed through it, that the centre of gravity of the whole sector CAB will coincide with the centre of gravity G of this circular arc DE.

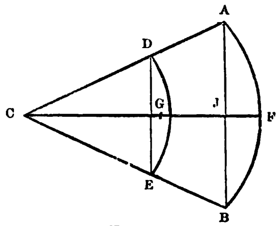


Fig. 56.

Now we have, radius CD$=\frac{2}{3}$ radius CA, chord DE$=\frac{2}{3}$ chord AB, and arc DE$=\frac{2}{3}$ arc AB; therefore, by substitution in eq. (1), we get,

$$\text{CG}=\frac{2}{3}\cdot\frac{\text{radius CA}\times\text{chord AB}}{\text{arc AB}}\ldots(2),$$

which gives the distance of the centre of gravity of the sector from the centre measured on the line CF bisecting the arc.

Let CA$=$R, angle ACB$=2\theta$; then chord AB$=2$R $\sin\theta$, and arc AB$=2$Rθ; substituting in eq. (2), we get,

$$\text{CG}=\frac{2}{3}\cdot\frac{\text{R}\sin\theta}{\theta}\ldots(3).$$

(2.) To find the centre of gravity of a circular ring ABFE.

Draw CE bisecting the arc AB, and on this line, let G be the centre of gravity of the sector ABC, G_1 that of CED, and g that of the circular ring ABED.

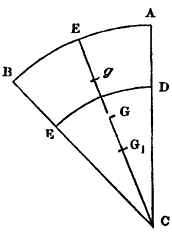


Fig. 57.

From the principle of moments, we have,

$$\text{sector CAB}\times\text{CG}=\text{sector CDE}\times\text{CG}_1$$
$$+\text{ring ABED}\times\text{C}g;$$

$$\therefore\ \text{C}g=\frac{\text{sector CAB}\times\text{CG}-\text{sector CDE}\times\text{CG}_1}{\text{ring ABED}}\ldots(4).$$

Let CA$=$R, CD$=r$, $\angle$ACB$=2\theta$; then,

$$\text{sector CAB}=\text{R}^2\theta,\ \text{sector CDE}=r^2\theta,$$

ring ABED$=\text{R}^2\theta-r^2\theta$, CG is given in eq. (3), and $\text{CG}_1=\frac{2}{3}\cdot\frac{r\sin\theta}{\theta}$;

hence, we find by substitution in eq. (4), and reducing,

$$\text{C}g=\frac{2}{3}\cdot\frac{\text{R}^3-r^3}{\text{R}^2-r^2}\cdot\frac{\sin\theta}{\theta}\ldots(5).$$

which gives the distance of the centre of gravity of the ring from the centre C.

(3.) To find the centre of gravity of the segment AFB (see *fig.* 56.).

Taking C as the centre of moments, we have,

Momt. sector CAFB$=$momt. triangle CAB$+$momt. segt. AFB.

$$\text{R}^2\theta\times\frac{2}{3}\frac{\text{R}\sin\theta}{\theta}=\text{R}^2\sin\theta\cos\theta\times\frac{2}{3}\text{R}\cos\theta+(\text{R}^2\theta-\text{R}^2\sin\theta\cos\theta)\text{X},$$

$$\therefore\ \text{X}=\frac{2}{3}\cdot\frac{\text{R}\sin^3\theta}{\theta-\sin\theta\cos\theta}\ldots(6).$$

109. To find an approximate formula for the height of the centre of gravity of any solid, having its parallel sections similar to one another.

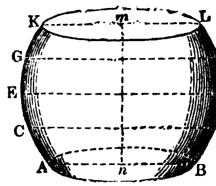


Fig. 58.

Let the perpendicular height be divided into n equidistant sections. Put $h=$ the perpendicular height of each section; $k_0^2, k_1^2, \ldots, k_n^2=$ the area of the successive sections; then, regarding each slice of the solid as a truncated pyramid; we have, by eq. (2), Art. **97**.

Moment of the nth or last slice,

$$=\left\{(n-1)h+\frac{h}{4}\cdot\frac{k_{n-1}^2+2k_{n-1}k_n+3k_n^2}{k_{n-1}^2+k_{n-1}k_n+k_n^2}\right\}\frac{h}{3}(k_{n-1}^2+k_{n-1}k_n+k_n^2)$$

$$=\frac{h^2}{12}\{(4n-3)k_{n-1}^2+(4n-2)k_{n-1}k_n+(4n-1)k_n^2\}.$$

Now this moment is a type of all the others; hence, therefore, by taking n successively equal to 1, 2, 3, . . ., n, we obtain a series of terms for the moments of the 1st, 2nd, 3rd, . . ., nth slices composing the solid, the sum of which will be equal to the moment of the whole solid; but the moment of the whole solid will be expressed by $y \times$ whole solid;

$$\therefore\ y=\frac{\text{sum of the moments of all the slices}}{\text{whole solid}};$$

hence, we find, by summing up and reducing,

$$y=\frac{h}{4}[k_0^2+(4n-1)k_n^2+8\{k_1^2+2k_2^2+\ldots+(n-1)k_{n-1}^2\}$$
$$+2\{k_0k_1+3k_1k_2+\ldots+(2n-1)k_{n-1}k_n\}]$$
$$\div\{k_0^2+k_n^2+2(k_1^2+\ldots+k_{n-1}^2)+k_0k_1+\ldots+k_{n-1}k_n\}.$$

Example. If $n=3$, then

$$y=\frac{h}{4}[k_0^2+11k_3^2+8(k_1^2+2k_2^2)+2(k_0k_1+3k_1k_2+5k_2k_3)]$$
$$\div\{k_0^2+k_3^2+2(k_1^2+k_2^2)+k_0k_1+k_1k_2+k_2k_3\}$$

If the solid be formed by the revolution of the curve LHDB about the axis nm, then $k_0^2=\pi r^2$, &c.; and hence, the k's in the foregoing formula must be taken to represent the radii of the sections.

This theorem is given here for the first time.

Work in moving the Centre of Gravity of Bodies.

110. If in Art. **72.**, the different pressures be regarded as the weights of the parts of a system of bodies, and the resultant X the weight of the whole system acting in its centre of gravity, then eq. (1) may be interpreted as follows: THE SUM OF THE WORKS DONE IN MOVING THE PARTS OF THE SYSTEM, THROUGH ANY VERTICAL SPACES, IS EQUAL TO THE WORK REQUISITE TO MOVE THE WHOLE SYSTEM OVER THE VERTICAL SPACE THROUGH WHICH THE CENTRE OF GRAVITY OF THE WHOLE SYSTEM IS MOVED.

This proposition is important in its application to industrial mechanics: thus, for instance, it enables us readily to find the work required in raising the materials of any structure; in pumping water from a well into a cistern; or in transferring materials, having a given form, from one place to another.

111. In like manner Art. **76.**, shows that WHEN ANY HEAVY BODY, WHOSE PARTS ARE RIGIDLY CONNECTED, IS TURNED ROUND A FIXED CENTRE IN OPPOSITION TO GRAVITY, THEN THE WORK DONE IS EQUAL TO THE WORK REQUISITE TO RAISE THE WHOLE BODY OVER THE VERTICAL HEIGHT THROUGH WHICH ITS CENTRE OF GRAVITY IS RAISED.

For instance, let it be required to find the work requisite to overturn the body ABV, standing on the horizontal plane AB.

Let g be the centre of gravity of the body, and Abv its position when it is about to fall over, n being its centre of gravity in this

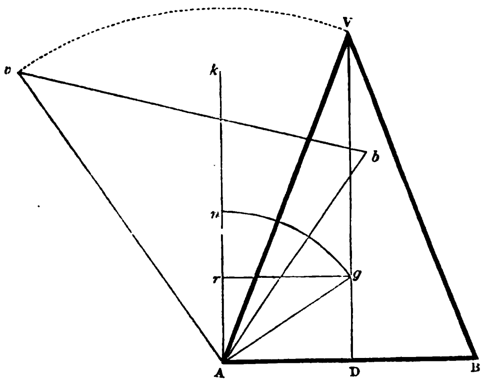


Fig. 59.

position. Now An must be a vertical line, and if gr be drawn parallel to AB, the centre of gravity of the body will have been elevated through the vertical space rn.

$\therefore$ Work in overturning the body=weight body $\times rn$.

Put $y=\text{D}g$ the height of the centre of gravity of the body; $x=\text{AD}$; and W=the weight of the body in lbs.; then $\text{A}g=\sqrt{x^2+y^2}$; $rn=\text{A}n-\text{A}r=\text{A}g-\text{D}g=\sqrt{x^2+y^2}-y$;

$\therefore$ Work in overturning the body$=\text{W}(\sqrt{x^2+y^2}-y)$ . . . (1.)

This work may be taken as a measure of the stability of a body; in this sense, the resulting formula is said to be an expression for the *dynamical stability* of the body.

112. *If a machine, the pieces of which move without friction, be in equilibrium in all positions, under the action of weights suspended from any parts of the machine; the common centre of gravity of the system (supposing motion to take place) will neither ascend nor descend.*

For since the system is in equilibrium in all positions, it follows that no work can be done by changing the relative position of the parts, that is to say, the aggregate amount of work of all the parts must be equal to zero, and consequently the common centre of gravity will neither ascend nor descend.

The example given at page 125. is a good illustration of this general principle.

Exercises for the Student.

Centre of Gravity.

1. If three equal heavy particles be connected so as to form a triangle, their centre of gravity coincides with that of the triangle. Required the proof.

2. If G be the centre of gravity of the triangle ABC, and GA, GB, GC be joined, then three forces which are proportional to GA, GB, GC, will maintain the point G in equilibrium.

3. Four bodies whose weights are 3, 4, 5, and 6 lbs., are places at the successive angles of a square whose side is 9 inches, show that the distance of the centre of gravity from the greatest weight is $\frac{1}{3}\sqrt{130}$ inches.

4. Three bodies, considered as points, are placed at the angles of a triangle; their weights are proportional to the opposite sides of the triangle; show that their centre of gravity is the centre of the inscribed circle.

5. If the sides of the triangle ABC be bisected in the points D, E, and F, and these points be joined, then the centre of the circle described within the triangle DEF is the centre of gravity of the

three sides of the triangle ABC considered as three uniform rods. Prove this by means of the property given in Problem 4.

6. Two isosceles triangles, ABC and ABD, are described upon the same base AB and on the same side of it; required the centre of gravity of the surface ACBD included between the sides of the triangles. *Ans.* Distance centre of gravity from the base $=\frac{1}{3}(a+b)$, where a and b are put for the perpendicular heights of the triangles.

7. If two circles touch each other internally; find the centre of gravity of the surface included between the two circles.

8. Triangles are described in a given circle upon a given chord as a base, the locus of their centres of gravity is an arc of a circle.

9. A square board is placed with one side on a horizontal table, how must the board be cut by a plane extending from one of its upper corners to meet the base, so that it may just be upon the point of falling?

10. A heavy triangular plate is suspended in a horizontal position by three threads attached to its corners; prove that the threads will all have the same tension. Also show that, if the threads are of the same strength, the best part of the plate to put a heavy weight is at the centre of gravity of the plate.

11. A regular hexagonal prism is placed upon an inclined plane with its end faces vertical, what must be the inclination of the plane so that the prism may just trundle down the plane? *Ans.* 30°.

12. A regular polygon just trundles down an inclined plane whose inclination is 10°; how many sides has the polygon?

Ans. 18.

13. If G be the centre of gravity of the sector CAB (see *fig.* 57., Art. **108.**), G_1 that of CDE, and g that of the ring ABED; prove that $R^2 \times Gg = r^2 \times G_1 g$.

14. The distance of the centre of gravity of a semicircle is at $\frac{4}{3} \cdot \frac{r}{\pi}$ from the diameter, r being the radius of the circle.

15. Construct for the centre of gravity of two of the sides of a triangle.

16. A side and the centre of gravity of a triangle is given, to construct it.

17. On a horizontal base to construct a triangle, having a given perpendicular height, such that the vertical line through its centre of gravity shall pass through one extremity of the base.

18. One-fourth of a parallelogram is cut off by a line parallel to one of its diagonals. Find the centre of gravity of the remainder.

19. From a square ABCD, whose diagonals intersect in N, the triangle ANB is cut out; find the centre of gravity of the remainder.

20. If two bodies approach each other with velocities which are inversely as their weights, their common centre of gravity will remain stationary.

21. Two bodies A and B move in the same straight and in the same direction with the velocities V and v respective ; to find the velocity V^1 of their common centre of gravity.

Let a, b, c, be the distances of the bodies A and B, and of their centre of gravity, measured from any fixed point in the line of motion, at any instant; and let a_1, b_1, c_1, represent the same quantities after the bodies have moved for t seconds.

Now by the equality of moments, we have

$$(A+B)c=Aa+Bb,$$

$$\text{and } (A+B)c_1=Aa_1+Bb_1,$$

$$\therefore (A+B)(c_1-c)=A(a_1-a)+B(b_1-b);$$

but $c_1-c=tV^1$, $a_1-a=tV$, and $b_1-b=tv$; hence by substitution and reduction, we get

$$V^1=\frac{A.V+B.v}{A+B} \quad \ldots (1).$$

22. When the bodies move in opposite directions, show that $V^1=\frac{A.V-B.v}{A+B}$.

23. When the bodies are equal, show that the velocity of the centre of gravity is equal to the mean velocity of the two bodies.

24. Two bodies A and B move uniformly along the adjacent sides of a rectangle; required the locus of their common centre of gravity, when they start from the same angle at the same instant A along the one side and B along the other.

CHAP. VI.

MECHANICAL POWERS.

113. THE mechanical powers are certain simple machines or instruments commonly used for raising heavy weights; it is customary to call the moving pressure the *power*, and the body which is moved the *weight*. The advantage gained is the number of times

that the weight is greater than the power. There are six mechanical powers,—the Lever, the Wheel and Axle, the Pulley, the Inclined Plane, the Wedge, and the Screw.

The Lever.

114. A lever is a rigid rod, moveable about a fixed point, called the fulcrum or centre of motion. In *fig.* 59., F is the fulcrum or centre of motion, W is the weight or resistance to be moved, and P is the power.

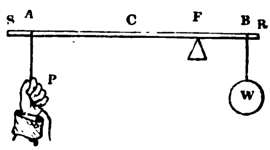


Fig. 60.

Levers are divided into three kinds according to the relative positions of power and weight with respect to the fulcrum.

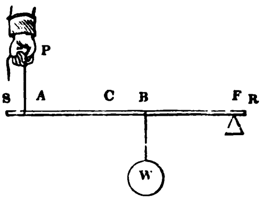


Fig. 61.

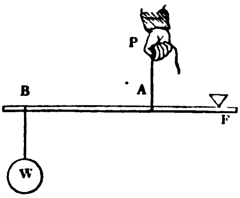


Fig. 62.

Fig. 60. represents a lever of the first kind; *fig.* 61. a lever of the second kind; and *fig.* 62. of the third kind.

The equation of equilibrium, demonstrated in Art. **78.**, applies to all these cases of the lever; thus we have

$$P \times AF = W \times BF \ldots (1),$$

and for the mechanical advantage, we have

$$\frac{W}{P} = \frac{AF}{BF} \ldots (2).$$

It will be observed that these equations hold true for all directions of the forces P and W, provided that they act parallel to each other.

115. When a series of weights, P_1, P_2, &c., W_1, W_2, &c., are suspended from a lever, we have on the principle of the equality of moments, Art. **81.**, for the condition of equilibrium.

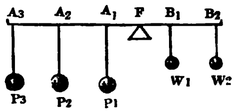


Fig. 63.

$$P_1 \times A_1F + P_2 \times A_2F + \&c. =$$
$$W_1 \times B_1F + W_2 \times B_2F + \&c.$$

Compound Lever.

116. In fig. 64. AC, B_1Q, D_1R, represent a series of levers, turning upon the fixed centres of motion B, Q, R. Let P be the power applied at A, and W the weight acting at E. Put R for the force produced at B or B_1, and R_1 the force produced at D or D_1; then in the lever AC, we have

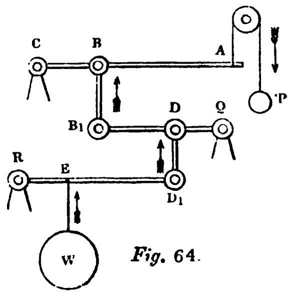


Fig. 64.

$$P \times AC = R \times BC;$$

in the lever B_1Q,

$$R \times B_1Q = R_1 \times DQ;$$

and in the lever D_1R,

$$R_1 \times D_1R = W \times ER;$$

multiplying these equations together, and striking out the factors common to both sides of the equation, we find

$$P \times AC \times B_1Q \times D_1R = W \times BC \times DQ \times ER,$$

and for the mechanical advantage

$$\frac{W}{P} = \frac{AC \times B_1Q \times D_1R}{BC \times DQ \times ER} \ldots (1).$$

117. Prop. *To find the conditions of equilibrium, &c. when the direction of the forces are not perpendicular to the lever.*

Let AB be the lever, C its fulcrum, SB the direction of the force W, and TA the direction of P. From C let fall the perpendiculars CQ and CR upon the respective directions of the forces P and W; then we have by the equality of moments

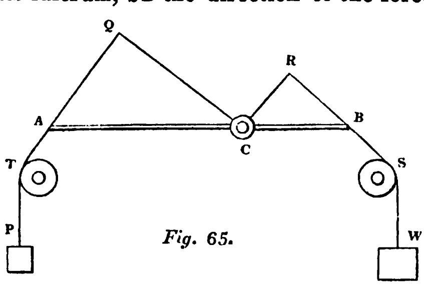


Fig. 65.

$P \times CQ = W \times CR \ldots (1).$

Let $\alpha = \angle CAQ$, and $\beta = \angle CBR$; then

$$CQ = AC \sin \alpha, \text{ and } CR = BC \sin \beta;$$

substituting these values in eq. (1), we get

$$P \times AC \sin \alpha = W \times BC \sin \beta \ldots (2).$$

To find the pressure on the fulcrum and its direction; let the directions of the pressures P and W intersect in D; join D and C; then DC will be direction of the resultant of P and W, for it must be destroyed by the reaction of the fulcrum. Put R for this resultant, then by Art. 58., we have

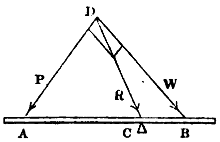


Fig. 66.

$$R^2 = P^2 + W^2 + 2P \cdot W \cdot \cos ADB;$$

but $ADB = 180 - (\alpha + \beta)$,

$$\therefore R^2 = P^2 + W^2 - 2P \cdot W \cdot \cos(\alpha + \beta) \ldots (3).$$

whence the pressure, R, upon the fulcrum is determined.

To find the direction of the pressure upon the fulcrum: let $\phi = \angle DCA$. Taking R as the reaction of the fulcrum opposite to the resultant of P and W, and resolving the pressures P, W, and R horizontally, we get

$$P \text{ resolved horizontally} = P \cos \alpha,$$
$$W \quad ,, \qquad ,, \qquad = W \cos \beta,$$
$$R \quad ,, \qquad ,, \qquad = R \cos \phi;$$

but as the lever has no motion, these forces mutually destroy one another, or in this case, the difference of the resolved forces of P and W must be equal to the resolved force of R; hence we get

$$W \cos \beta - P \cos \alpha = R \cos \phi.$$

In like manner resolving the forces perpendicularly, we get

$$P \sin \alpha + Q \sin \beta = R \sin \phi;$$

hence we have by division

$$\tan \phi = \frac{P \sin \alpha + Q \sin \beta}{W \cos \beta - P \cos \alpha} \ldots (4).$$

which gives the inclination of the resultant.

Similar expressions may be found when the lever is of the second kind.

The Lever when its weight is taken into account.

118. Here the weight of the lever acts in its centre of gravity.

Let C be the centre of gravity of the lever SR (see *figs.* 60. and 61.), then we have by the equality of moments:

For the first kind of lever (see *fig.* 60.),

$$P \times AF + \text{wt. lever} \times CF = W \times BF;$$

and for the second kind of lever (see *fig.* 61.)

$$P \times AF = W \times BF + \text{wt. lever} \times CF.$$

Put $AF=a$, $BF=b$, $SF=q$, $RF=r$, and the weight of a unit of length of the lever$=w$; then wt. lever$=w \times SR = w(q+r)$, and $CF=SF-SC=q-\frac{1}{2}(q+r)=\frac{1}{2}(q-r)$. Hence we find by substituting in the preceding equations of moments;

$$Pa+\tfrac{1}{2}w(q^2-r^2)=Wb \ldots (1),$$

which is the equation of equilibrium for the lever of the first kind;

$$Pa=Wb+\tfrac{1}{2}w(q^2-r^2) \ldots (2),$$

which is the equation for the lever of the second kind.

In eq. (2), if P is applied at the extremity S and the fulcrum F is at the extremity R, then $q=a$, and $r=0$; and in this case we have

$$Pa=Wb+\tfrac{1}{2}wa^2,$$

$$\text{and } P=\frac{Wb}{a}+\tfrac{1}{2}wa \ldots (3).^*$$

119. If we suppose SR (see *fig.* 61.) to be a beam supported by props at F and A, and loaded with a weight W at B; then the value of P determined from eq. (3) will give the pressure on the prop at A.

To graduate the Lever of the Safety Valve.

120. Let RB represent the lever; B its centre of motion; C its centre of gravity; v the valve; and L the load.

For the condition of equilibrium, we have by the equality of moments

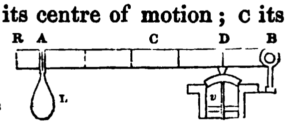


Fig. 67.

$$L \times AB + \text{wt. lever} \times CB$$

$$= \{\text{pressure steam on valve} - \text{wt. valve}\} \times DB.$$

* Prop. *To find* a *when* P *is a minimum.*

By differentiation, we have

$$\frac{dP}{da} = -\frac{Wb}{a^2}+\tfrac{1}{2}w=0;$$

$$\therefore a^2=\frac{2Wb}{w},$$

$$\therefore a=\sqrt{\frac{2Wb}{w}}.$$

Put q = RB, the length of the lever; then $\frac{1}{2}q$ = CB;
„ a = AB, the distance of the load from B;
„ b = DB, the distance of the head of the valve from B;
„ k = the section of the valve in sq. inches;
„ P = the pressure of the steam in lbs. per sq. inch;
„ w_1 = the weight of the valve;
„ w = the weight of each linear inch of the lever.

Hence we have,

wt. lever $= qw$; pressure steam on valve $= k$P;

and substituting in the equation of moments we get for the conditions of equilibrium

$$\text{L}a + \tfrac{1}{2}q^2w = (k\text{P} - w_1)b \ldots (1).$$

(1). To determine the load so as to produce a given maximum pressure of the steam.

In this case the load must be placed at the extremity of the lever, and therefore $a=q$, hence we have from eq. (1)

$$\text{L} = (k\text{P} - w_1)\frac{b}{q} - \tfrac{1}{2}qw \ldots (2).$$

(2). To determine the point A on the lever so that the load L may produce any given pressure P of the steam.

Hence from eq. (1) we have

$$a = \frac{(k\text{P} - w_1)b - \frac{1}{2}q^2w}{\text{L}} \ldots (3);$$

where it is to be observed that L is given by eq. (2).

Example. Let BR $= q = 20$ inches; DB $= b = 2$ inches; $k = 6$ sq. in.; P $= 50$ lbs. the maximum pressure of the steam; $w = \frac{1}{4}$ lb.; $w_1 = 4$ lbs.

To find L. By formula (2)

$$\begin{aligned}\text{L} &= (6 \times 50 - 4)\tfrac{2}{20} - \tfrac{1}{2} \times 20 \times \tfrac{1}{4} \\ &= 27{\cdot}1 \text{ lbs.}\end{aligned}$$

To find a or the point on the lever which will give, for example, a pressure of 30 lbs. of the steam. Here in eq. (3), we have L $= 27{\cdot}1$, and P $= 30$,

$$\therefore\ a = \frac{(6 \times 30 - 4)\,2 - \frac{1}{2} \times 20^2 \times \frac{1}{4}}{27{\cdot}1} = 11{\cdot}1 \text{ in.},$$

which gives the distance of the point from B as required. In like manner the point corresponding to any other pressure may be de-

termined; but the following method of graduation is much more elegant and expeditious.

121. *New Method.* From eq. (1) we get

$$a+\frac{bw_1+\frac{1}{2}q^2w}{\text{L}}=\frac{bk}{\text{L}}.\text{P};$$

$$\text{or putting } e=\frac{bk}{\text{L}},\text{ and } c=\frac{bw_1+\frac{1}{2}q^2w}{\text{L}},$$

this equation becomes

$$a+c=e\text{P}\ldots(4).$$

Let a_n be the distance of the load from B, when the pressure is nP, and a_{n+1} the distance when the pressure is $(n+1)$P, then we have by eq. (4)

$$a_n+c=en\text{P},$$

$$\text{and } a_{n+1}+c=e(n+1)\text{P},$$

$$\therefore\ a_{n+1}-a_n=e\text{P}$$

$$=\frac{bk}{\text{L}}.\text{P}\ldots(5),$$

which expresses the interval on the lever between the pressures nP and $(n+1)$ P. But this expression, being independent of n, shows that the intervals on the lever between the pressures P, 2P, 3P, &c. are equal to one another. Having determined the point on the lever corresponding to any proposed pressure of the steam by either of the first two equations, the interval on the lever corresponding to any given interval of pressure may be found from eq. (5).

Taking the data in the foregoing example, let it be required to find the interval on the lever corresponding to every 5 lbs. pressure of the steam.

Here by eq. (5), we have

$$\text{The interval of 5 lbs.}=\frac{bk}{\text{L}}.\text{P}$$

$$=\frac{2\times 6}{27{\cdot}1}\times 5=2{\cdot}214\text{ in.}$$

Now we have already found that the load placed at the extremity of the lever produces a pressure of 50 lbs.; therefore, by marking off 2·214 in. along the lever from its extremity, we shall find the points corresponding to the pressures of 50 lbs., 45 lbs., 40 lbs., 35 lbs., &c.

122. To determine the true weight of a body by means of a false balance.

In a false balance the arms of the beam are of unequal length.

Let a = the length of the long arm;
„ b = „ „ short arm;
„ W = the true weight of the body;
„ P = the weight of the body when placed in the scale attached to the long arm;
„ p = its weight when placed in opposite scale.

Now as W is balanced in both cases, we have by the principle of the lever,

$$W \times a = P \times b,$$

$$\text{and } W \times b = p \times a,$$

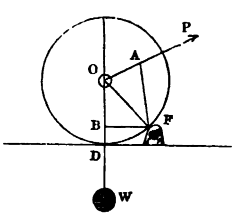
multiplying these equations together

$$W^2 ab = Ppab,$$

$$\therefore \; W = \sqrt{Pp},$$

which give the true weight required.

123. To find the force necessary to draw a carriage over a small obstacle.



Let O be the centre of the wheel, F the obstacle, W the weight of the carriage acting through the axis O, OP the direction of the traction P. From F draw FA, FB, perpendicular to the direction of the forces P and W. Now, in order that the wheel may turn over the obstacle, it must turn round F as a centre of motion; hence we have, by the quality of moments, Art. **117.**,

$$P \times FA = W \times FB,$$

$$\therefore \; P = \frac{FB}{FA} \cdot W.$$

Here P will be a minimum when FA is a maximum, that is, when FA equals FO or when OP is perpendicular to FO.

Put r=OF the radius of the wheel, and h=DB the weight of the obstacle, and $\alpha = \angle$ FOA; then the preceding condition becomes

$$P = \frac{\sqrt{r^2 - (r-h)^2}}{r \sin \alpha} \cdot W$$

$$= \frac{W}{\sin \alpha} \cdot \sqrt{\frac{h}{r}\left(2 - \frac{h}{r}\right)} = \frac{W\sqrt{2h}}{\sin \alpha} \cdot \frac{1}{\sqrt{r}}, \text{ nearly,}$$

because h is small as compared with r. Hence it appears that the power necessary to draw the carriage over an obstacle (other things being the same) varies inversely as the square root of the radius of the wheel.

124. ***The principle of the equality of work as applied to the lever.***

(1). Let PW be a straight lever acted upon by the pressures P and W applied perpendicularly to the lever. Now when the lever is moved to the position pw, the pressure P has moved over the arc Pp, while W has moved over Ww;

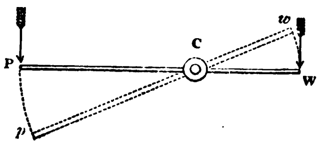


Fig. 68.

$\therefore$ Work of P$=$P$\times$arc Pp, and work of W$=$W$\times$arc Ww; but by the principle of the lever, Art. **114.**,

$$\frac{P}{W}=\frac{CW}{CP}=\frac{\text{arc } Ww}{\text{arc } Pp},$$

$$\therefore\ P\times\text{arc } Pp=W\times\text{arc } Ww,$$

that is, work of P$=$work of W.

(2). Let the bent lever ACB turn upon its centre C, by the action of the forces P and W applied obliquely to the arms CA and CB.

Let CQ and CR be perpendiculars from the centre of motion C, upon the directions of the forces, and let the lever ACB be moved into the new position acb very near to the first. From a let fall ae perpendicular to PA produced; then while the extremity A of the arm CA describes the arc Aa, the force P will have moved, in the line of its action, through Ae. Now when the change of position is indefinitely small, the circular arc Aa becomes a straight line perpendicular to CA, and Aea is a right-angled triangle; moreover, if Aa be the space moved over by the extremity of the arm CA, the space moved over by P, in the direction Pe of its action, will be Ae.

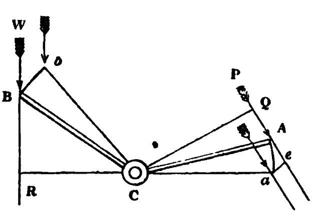


Fig. 69.

Because $\angle$CAa is a right angle, the $\angle a$Ae is equal to the $\angle$ACQ, and the triangles Aea and ACQ are equiangular,

$$\therefore\ \frac{Aa}{CA}=\frac{Ae}{CQ}=\frac{\text{space moved by P}}{CQ}.$$

In like manner we have

$$\frac{Bb}{CB}=\frac{\text{space moved by W}}{CR};$$

but since $\angle ACa=\angle BCb$, we have

$$\frac{Bb}{Aa}=\frac{C}{CA}, \text{ or } \frac{Aa}{CA}=\frac{Bb}{CB};$$

$$\therefore \frac{\text{space moved by P}}{CQ}=\frac{\text{space moved by W}}{CR};$$

now by the equality of moments, Art. **117.**, we have

$$P\times CQ=W\times CR;$$

multiplying these equations together

$$P\times\text{space moved by }P=W\times\text{space moved by }W,$$

that is, work of P=work of W.

Now as this equality will be true for any number of small motions that may be given in succession to the lever, provided only the forces are constantly in equilibrium, it follows that the equality will hold true for any definite motions that may be given.

Wheel and Axle.

125. This mechanical power is only another form of the lever, where the power is made to act without intermission; in its most simple form, it consists of a horizontal axle A and large wheel R, which turn upon two pivots supported in gudgeons. A cord wrapping round the axle A sustains the weight W, and another cord wrapping round the wheel R, in a contrary direction, sustains the power P. These forces always act in the direction of a tangent to the circle. Here the leverage of the power is the radius of the wheel, and the leverage of the weight is the radius of the axle; hence we have

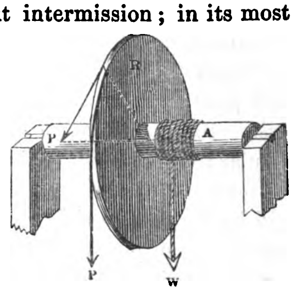


Fig. 70.

$$P\times\text{rad. wheel}=W\times\text{rad. axle},$$

and for the advantage gained,

$$\frac{W}{P}=\frac{\text{rad. wheel}}{\text{rad. axle}}\ldots(1).$$

126. This figure represents a combination of wheels and axles. F is a wheel, to which the power P is applied, and BC its axle turning upon a common axis; AD another wheel, with its axle E sustaining the weight W.

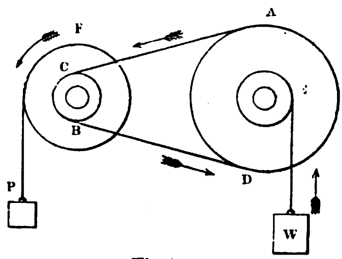


Fig. 71.

Let R = the radius of the wheel F, and r = the radius of its axle BC; R_1 = the radius of the wheel AD, and r_1 = the radius of its axle E; Q = the force of tension produced on the cord CA; then we have for the conditions of equilibrium

$$P \times R = Q \times R, \text{ and } Q \times R_1 = W \times r_1,$$

and multiplying these equations together,

$$P \times R \times Q \times R_1 = Q \times r \times W \times r_1,$$

$$\therefore\ P \times R \times R_1 = W \times r \times r_1 \ldots (2);$$

thus it appears that the power multiplied by the radii of the wheels is equal to the weight multiplied by the radii of the axles.

127. In the compound wheel and axle, represented in *fig.* 72. let R = the radius of the large axle A; r = the radius of the small axle D; and l = the length of the handle PO. Now as the weight W is suspended by the two cords CD and AB, they will each have a tension of $\frac{1}{2}$ W, hence we have by the equality of moments

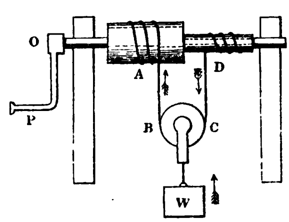


Fig. 72.

$$P \times l + \tfrac{1}{2} W \times r = \tfrac{1}{2} W \times R,$$

$$\therefore\ \frac{W}{P} = \frac{2l}{R - r} \ldots (1),$$

which is the expression for the advantage gained. This evidently increases with the smallness of the *difference* of the radii of the axles.

Cogged or Toothed Wheels.

128. Let D be a cogged wheel turning upon the same axis as the wheel C; Q another cogged wheel, acted upon by the former,

and turning upon the same axis as the axle I. From the wheel C is suspended the weight P, and from the axle I the weight W; then while P descends, the wheel C and the cog D will be turned from right to left; but as every tooth in the cog D is being turned round, a corresponding tooth in the cog Q will be turned in the contrary direction, and thus the cord IW will be coiled up upon the axle I, and the weight W will be raised.

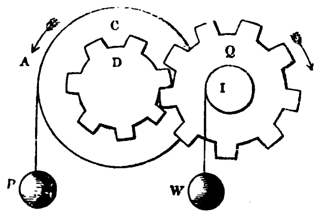


Fig. 73.

Let R = the radius of the wheel, A; r = the radius of the toothed wheel D; R_1 = the radius of the toothed wheel Q; r_1 = the radius of the axle I; then proceeding as in Art. **126.**, we have for the conditions of equilibrium

$$P \times R \times R_1 = W \times r \times r_1 \ldots (1),$$

and for the advantage gained,

$$\frac{W}{P} = \frac{R \cdot R_1}{r \cdot r_1} \ldots (2).$$

Let n = the number of teeth in D, and n_1 = the number in Q; then since the number of teeth in the wheels are proportional to their radii, we should have $\frac{R_1}{r} = \frac{n_1}{n}$; hence eq. (2) becomes

$$\frac{W}{P} = \frac{R \cdot n_1}{r_1 \cdot n} \ldots (3).$$

The principle of the equality of work applied to the wheel and axle.

129. As an example, let us take the combination of wheels and axles described in Art. **128.**, see *fig.* 73.

Let the axle I, with its toothed wheel Q, turn round once; then adopting the notation of Art. **128.**, we have

$$\text{No. turns of A with its cogged wheel D} = \frac{\text{No. teeth in Q}}{\text{No. teeth in D}}$$

$$= \frac{\text{circum. Q}}{\text{circum. D}} = \frac{R_1}{r}.$$

$$\therefore \text{ Space moved over by } W = 2\pi r_1,$$

$$\text{and } \quad ,, \quad ,, \quad ,, \text{ by } P = 2\pi R \times \frac{R_1}{r}.$$

$$\therefore \text{ Work } W = W \times 2\pi r_1,$$

$$\text{and Work } P = P \times \frac{2\pi R R_1}{r}.$$

Now by multiplying each side of eq. (1), Art. **128.**, by $\frac{2\pi}{r}$, we get

$$P \times \frac{2\pi R R_1}{r} = W \times 2\pi r_1;$$

that is, Work P = work W.

If the weights, in the foregoing problems, be moved, their common centre of gravity will neither ascend nor descend.

130. Suppose the weights to be placed so that their centres of gravity shall lie in the same horizontal line, then their common centre of gravity will also lie in this line. Now when they are moved from this position, we have, by the equality of work just established,

$$P \times \text{space moved over by } P = W \times \text{space moved over by } W;$$

hence it follows, see Art. **92.**, that the common centre of gravity of the two weights, in this new position, must lie in the horizontal line in which they were at first placed.

The same principle may be demonstrated for any other combination of wheels.

The Pulley.

131. A pulley consists of a small wheel having a round groove made in its circumference for receiving a cord; this wheel is placed in a frame called the block, and turns on an axis resting between the sides of the block. The pulley is said to be fixed or moveable, according as its block is fixed or moveable. There are various combinations of pulleys; in all of them a force called the power (P) is applied to the first string, and this sustains another force, called the weight (W), applied to the last string.

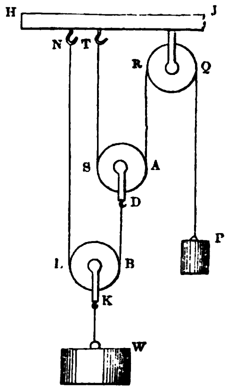


Fig. 74.

132. In the annexed system, there are two moveable pulleys, A and B, and one fixed pulley Q. Here the string to which the power is attached passes over the fixed pulley Q, then round the moveable pulley A, and has its extremity fixed at T. Another string is attached to the block of the pulley A, then passes round the moveable pulley B, and has its extremity fixed at N.

Here PQRAT, being a continuous cord, will be stretched equally throughout the whole of its length; and the cords AR and ST will each have a tension of P lbs.; and therefore a weight of 2P lbs. must be suspended from D; in like manner, since

DBLN is a continuous cord, LN and BD will have the same tension, that is, each of them will have a tension of 2P lbs.; and therefore a weight of twice 2P lbs., or 4P lbs., must be suspended from K; that is to say, in the system represented in *fig.* 74., we have W=4P. Thus it appears that every successive moveable pulley doubles the weight that would be sustained by the preceding pulley. If n therefore, be put for the number of moveable pulleys, we have

$$W=2^n P \ldots (1).$$

133. Let us now take the weights of the pulleys into account; and, for this purpose, put w=the weight of each moveable pulley; then

Weight suspended from D, or tension cord DBLN$=2P-w$,
" " K, " " $=2(2P-w)-w$;

hence we have in this case,

$$W=2(2P-w)-w$$
$$=2^2 P-(2^2-1)w \ldots (2).$$

And if there were n moveable pulleys, we should have,

$$W=2^n P-\{1+2+2^2+\ \ldots\ +2^{n-1}\} w$$
$$=2^n P-(2^n-1)w \ldots (3).$$

134. In this system (see *fig.* 75.) a single or continuous cord passes round the wheels, therefore every portion of the cord must have the same tension; but W hangs by six cords, therefore each cord will carry one-sixth of the weight W, and consequently the power P must also be one-sixth of W; that is, W=6P.

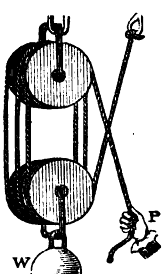


Fig. 75.

For other systems of pulleys, see the Author's "Elements of Mechanism," and "Exercises on Mechanics."

135. To find the relation of P and W in a single moveable pulley A, when the cords AC and AB are not parallel to each other.

Here as the cord PCAB is continuous, it must have the same tension in every part, therefore the cords AC and AB must each have a tension of P lbs., and the vertical line AK, in which the weight W acts, must bisect the angle CAB. Put $\alpha=\angle$PCA$=\angle$CAK; then the force P, acting by AC, resolved in the vertical direction AK is equal to P cos α, and the force P acting by AB, when thus re-

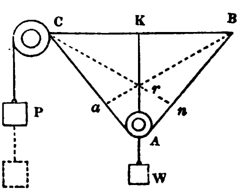


Fig. 76.

solved, will produce the same result. But W, acting vertically, will be equal to the sum of these resolved forces; hence we have

$$W=2P\cos\alpha \ldots (1),$$

which is the equation of equilibrium.

The Principle of the Equality of Work applied to the Pulley.

136. As an example, let us take the system of pulleys represented in *fig.* 74. See Art. **132.**

Let W, with its pulley B, ascend h feet; then, because of the double cord, LN and BD, the pulley A will ascend $2h$ feet, and therefore the cords ST and AR will each be shortened $2h$ feet, and consequently P will descend $4h$ feet; hence we have,

$$\text{Work } P=P\times 4h=4P\times h;$$

$$\text{Work W and the blocks}=Wh+wh+w\times 2h$$
$$=(W+3w)\times h,$$

but by eq. (2), Art. **133.**,

$$4P=W+3w;$$

$$\therefore\ 4P\times h=(W+3w)\times h;$$

that is,

$$\text{Work } P=\text{work W and the blocks.}$$

137. *If the weights in the foregoing problem be moved, their common centre of gravity will neither ascend nor descend.*

We shall here show this to be true, independently of the principle of work. See Art. **112.**

Assume HJ as the axis of moments; put q for the distance of W from HJ, b for the distance of the pulley B, a for that of A, k for the distance of P, and X for the distance of the common centre of gravity. Suppose W to be raised h feet, then B will rise h ft; A will rise $2h$ ft.; and P will descend $4h$ ft. Let X_1 be put for the distance of common centre of gravity in this position, and M for the weight of the whole mass; then we have, by the equality of moments,

$$X\times M=Wq+wb+wa+Pk,$$
$$X_1\times M=W(q-h)+w(b-h)+w(a-2h)+P(k+4h)$$
$$=Wq+wb+wa+Pk+4P\times h-(W+3w)\times h$$
$$=X\times M+0;$$

$$\therefore\ X_1\times M=X\times M,\text{ or } X_1=X;$$

that is to say, the distance of the centre of gravity from HJ remains the same.

THE INCLINED PLANE.

138. When a heavy body rests upon a hard horizontal surface, the force of gravity, acting vertically upon the body, produces a pressure perpendicular to the plane; in this case the whole weight of the body is expended in pressing upon the surface of the plane. But if the plane be inclined to the horizon, the force of gravity, acting vertically upon the body, will give rise to two forces, one of which acts perpendicular to the plane, and thereby produces a pressure upon it; the other acts parallel to the plane, and tends to move the body down it.

139. To determine the perpendicular pressure which a body produces on an inclined plane, and also to determine the tendency which the body has to descend the plane, the friction of the plane being neglected.

Let a be the body placed on the inclined plane ABC. Put W= the weight of the body, P=the pressure necessary to keep the body from descending, R=the perpendicular pressure on the plane, α=the inclination of the plane, or the angle BAC. Draw the vertical line ac; take the units in ac equal to the units of weight W; draw ak perpendicular to AC, and ck parallel to it, and complete the parallelogram of pressures $akce$; then the vertical pressure produced by the weight of the body represented by ac is equivalent to the two pressures ae and ak, the pressure ae being the force which urges the body down the plane, and ak that which produces the perpendicular pressure on the plane.

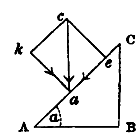


Fig. 77.

By the similar triangles aec and ABC, we have

$$\frac{ec}{ac}=\frac{AB}{AC};$$

but $ec=ak=R$, and $ac=W$,

$$\therefore \frac{R}{W}=\frac{AB}{AC};$$

$$\therefore R=W.\frac{AB}{AC}\ldots(1).$$

but $\frac{AB}{AC}=\cos\alpha$, hence we also have

$$R=W\cos\alpha\ldots(2),$$

which give the expressions for the perpendicular pressure of the body on the plane.

In like manner,

$$\frac{ac}{ae}=\frac{\text{AC}}{\text{BC}},$$

but $ae=\text{P}$, and $ac=\text{W}$,

$$\therefore \frac{\text{W}}{\text{P}}=\frac{\text{AC}}{\text{BC}}$$

$$=\frac{\text{length of the plane}}{\text{height of the plane}} \ldots (3).$$

$$\therefore \text{P}=\text{W}\cdot\frac{\text{BC}}{\text{AC}}$$

$$=\text{W}\frac{\text{height of the plane}}{\text{length of the plane}} \ldots (4),$$

but $\frac{\text{BC}}{\text{AC}}=\sin\alpha$, hence we also have

$$\text{P}=\text{W}\sin\alpha \ldots (5);$$

eq. (3) is the expression for the advantage gained; and (4) and (5) are the expressions for the tendency which the body has to descend the plane.

140. To find the conditions of equilibrium when a body, whose weight is W, is supported on an inclined plane by a force P whose direction makes a given angle β with the plane.

Let ac represent the weight of the body in direction and magnitude, and av that of the pressure P. Draw ak perpendicular to the plane, and complete the parallelograms of pressures shown in the figure; then by resolving the pressures, P and W, parallel and perpendicular to the plane, we have

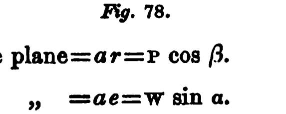


Fig. 78.

P resolved in the direction of the plane $=ar=\text{P}\cos\beta$.

W " " " $=ae=\text{W}\sin\alpha$.

P resolved perpendicular to the plane $=as=\text{P}\sin\beta$.

W " " " $=ak=\text{W}\cos\alpha$.

$\therefore$ Perpendicular pressure on the plane $=ak-as$,

$$\text{that is, } \text{R}=\text{W}\cos\alpha-\text{P}\sin\beta \ldots (1).$$

And when the body is in equilibrium the resolved pressures, *ar* and *ae*, must destroy each other, that is, they must be equal;

$$\therefore \text{P} \cos \beta - \text{W} \sin \alpha = 0,$$

$$\text{or } \frac{\text{W}}{\text{P}} = \frac{\cos \beta}{\sin \alpha} \ldots (2).$$

Eliminating P from eq. (1), we have

$$\text{R} = \text{W} \cos \alpha - \text{W} \frac{\sin \alpha \,.\, \sin \beta}{\cos \beta},$$

$$= \text{W} \frac{\cos (\alpha + \beta)}{\cos \beta} \ldots (3).$$

141. To find the conditions of equilibrium when a body is supported on an inclined plane by a pushing force (P) acting horizontally.

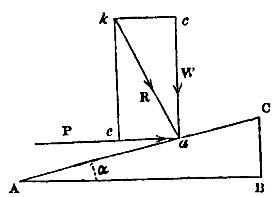


Fig. 79.

Let *ac* represent the pressure produced by the weight of the body in direction and magnitude; draw *ak* perpendicular to AC; and complete the parallelogram of pressures; then *ae* will represent the pressure P; and the pressures P and W, acting so that the body may neither move up the plane nor down the plane, will produce the resultant *ak* perpendicular to the plane. Because the line *ae* is parallel to AB, the triangles *akc* and ABC are similar, and

$$\therefore \frac{ac}{ck} = \frac{\text{AB}}{\text{CB}},$$

but $ac = \text{W}$, and $ck = ae = \text{P}$,

$$\therefore \frac{\text{W}}{\text{P}} = \frac{\text{AB}}{\text{CB}}, \text{ or } \frac{\text{base of the plane}}{\text{height of the plane}} \ldots (1);$$

but $\frac{\text{AB}}{\text{CB}} = \cot \alpha$,

$$\therefore \frac{\text{W}}{\text{P}} = \cot \alpha \ldots (2).$$

In like manner,

$$\frac{ac}{ak} = \frac{\text{AB}}{\text{AC}},$$

$$\text{that is, } \frac{\text{W}}{\text{R}} = \frac{\text{AB}}{\text{AC}}, \text{ or } \frac{\text{base of the plane}}{\text{length of the plane}} = \cos \alpha \ldots (3).$$

142. To find the conditions of equilibrium when two weights, P and W, rest on two inclined planes, AC and BC, having a common summit C, the weights being connected by a cord going over a pulley at C.

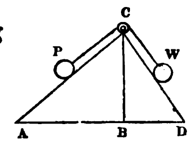


Fig. 80.

Here the tensions of the cords, CP and CW, must be equal, and consequently the tendencies of P and W down their respective planes must be equal; hence we have, by eq. (4), Art. **139.**,

$$\text{Tendency of P down CA} = \text{P} \cdot \frac{\text{BC}}{\text{AC}},$$

$$\text{Tendency of W down CD} = \text{W} \cdot \frac{\text{BC}}{\text{DC}},$$

$$\therefore \text{W} \cdot \frac{\text{BC}}{\text{DC}} = \text{P} \cdot \frac{\text{BC}}{\text{AC}},$$

$$\therefore \frac{\text{W}}{\text{P}} = \frac{\text{DC}}{\text{AC}} \ldots (1);$$

that is, the weights are proportional to the length of the planes on which they respectively rest.

The principle of the equality of work applied to the inclined plane.

143. First, let the cord, DW, by which the weight is sustained, be parallel to the plane AC. Now let W move from A to C; then W will have moved through the space BC in opposition to gravity; at the same time P will have descended a vertical space equal to AC; hence therefore

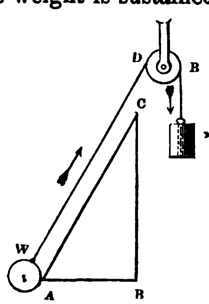


Fig. 81.

Work of W = W × BC, and work of P = P × AC

But we have by eq. (3), Art. **139.**, $\frac{\text{W}}{\text{P}} = \frac{\text{AC}}{\text{BC}}$,

or, P × AC = W × BC,

that is, work P = work W.

144. Supposing motion to be given to the weights, *their common centre of gravity neither ascends nor descends.*

Suppose the weights to be placed so that their centres of gravity shall be in the same horizontal line, then their common centre of gravity will also lie in this line; now when they are moved from this position, we have, by the equality of work just established,

P × vertical space moved over by P = W × vertical space moved over by W;

hence it follows by Art. **90.**, that the common centre of gravity of the two weights must lie in the horizontal line in which they were at first placed.

145. Second, suppose the cord DW to form any angle DWC (β) with the plane. Let W move over the very small space Ww; from D describe the arc wq, and draw Wk parallel to AB and wk perpendicular to it; then the cord WD will be shortened a space equal to Wq, and P will therefore descend a space equal to Wq; but W will ascend a vertical space equal to wk.

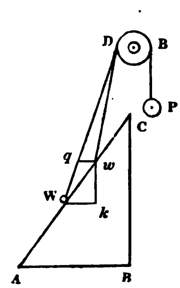


Fig. 82.

Now when Ww is indefinitely small, the arc wp becomes a straight line perpendicular to WD; hence we have

$$\text{Vertical space through which P descends} = \text{W}q$$
$$= \cos\beta \times \text{W}w;$$
$$\text{Vertical space through which W ascends} = wk$$
$$= \sin\alpha \times \text{W}w;$$

$$\therefore \text{ Work P} = \text{P} \cos\beta \times \text{W}w, \text{ and work W} = \text{W} \sin\alpha \times \text{W}w.$$

Now we have by eq. (2), Art. **140.**, for the equation of equilibrium

$$\text{P} \times \cos\beta = \text{W} \times \sin\alpha;$$
$$\therefore \text{ P} \cos\beta \times \text{W}w = \text{W} \sin\alpha \times \text{W}w;$$
$$\text{that is, work P} = \text{work W}.$$

THE MOVEABLE INCLINED PLANE, OR WEDGE.

146. Let ABC represent a moveable inclined plane, or wedge, sliding along the surface HR by the force of a pressure P applied to the back BC of the wedge in a direction parallel to HR; and let W be a heavy rod resting upon the inclined side AC, and constrained to move in a vertical direction.

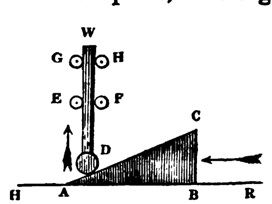


Fig. 83.

Here the weight W acts vertically, and the power P horizontally; hence we have, by eq. (1), Art. **141.**, for the conditions of equilibrium,

$$\frac{\text{W}}{\text{P}}=\frac{\text{AB}}{\text{CB}}=\frac{\text{length of the wedge}}{\text{thickness of the wedge}}\ldots(1).$$

147. If we suppose the direction, WDK, of the resistance to be at right angles to the inclined side AC, then we have, by resolving the force W into the horizontal direction HR,

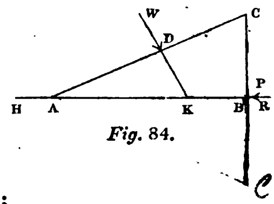


Fig. 84.

$$\text{W}\cos\angle\text{WKA}=\text{P};$$

$$\text{but}\cos\angle\text{WKA}=\sin\angle\text{BAC}=\frac{\text{BC}}{\text{AC}};$$

$$\therefore\ \text{W}\cdot\frac{\text{BC}}{\text{AC}}=\text{P};$$

$$\therefore\ \frac{\text{W}}{\text{P}}=\frac{\text{AC}}{\text{BC}}=\frac{\text{length of the side}}{\text{thickness of the back}}\ldots(2).$$ $\frac{W}{2P}=\frac{AC}{CC'}$

The Principle of Work applied to the Wedge.

148. Here, in *fig.* 83., whilst the power P moves over BA, the weight W moves over BC; hence we have

$$\text{Work P}=\text{P}\times\text{AB, and work W}=\text{W}\times\text{BC};$$

but by eq. (1), Art. **146.**, we have, for the equation of equilibrium,

$$\text{P}\times\text{AB}=\text{W}\times\text{CB};$$

that is, work P = work W.

The Screw.

149. In this simple machine, the power moves in a circle whose radius is the length of the lever, or arm of the screw, whilst the weight or resistance is moved in a right line, having the direction of the axis of the cylinder on which the threads of the screw are formed. A screw may be regarded as a moveable inclined plane formed upon the surface of the cylinder; for if we suppose one revolution of the thread to be unwrapped it will form an inclined plane, in which the circumference of the cylinder will be the length of the plane, and the distance between the threads the height of the plane.

Let *cname* be a spiral groove cut upon a cylinder after the

manner just described; CD the axis upon which the cylinder turns; AB a rod parallel to the axis CD, and having a pin or tooth *c* fitting the groove of the screw. Now when the handle CP is turned in the direction of the arrow, the pin *c*, with its rod AB, is moved towards the right; so that in one revolution the pin will have moved from *c* to *a*, the distance between the threads of the screw, and in the second revolution it will have moved from *a* to *e*, and so on. The rod AB will thus be moved in a rectilinear path, parallel to the axis CD.

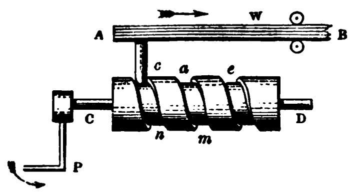


Fig. 85.

Let P=the power applied to the lever CP; W=the resistance acting in the direction AB; Q=the force equivalent to P which must be applied at the surface of the cylinder; r=the radius of the cylinder; a=the length of the lever CP; then we have, by the principle of the moveable inclined plane, Art. **146.**,

$$\frac{W}{Q}=\frac{\text{length of the plane}}{\text{distance between the threads}}$$

$$=\frac{\text{circum. cylinder}}{\text{distance between the threads}};$$

and by the principle of the lever,

$$\frac{Q}{P}=\frac{a}{r};$$

therefore by multiplication, we get

$$\frac{W}{P}=\frac{\frac{a}{r}\times\text{circum. cylinder}}{\text{distance between the threads}}$$

$$=\frac{\text{circum. described by P}}{\text{distance between the threads}}.$$

Now, in this investigation, we have supposed the whole of the resistance to be applied at one point of the thread of the screw; but it is obvious, that we may suppose this resistance to be spread over the thread without in the least affecting the result.

The Endless Screw.

150. When the threads of a screw are made to act upon the

teeth of a wheel, as in *fig.* 86., the mechanism is called the endless screw. Let a = the length of the lever AP; d = the distance between the threads of the screw; R_1, r_1 = the radii of the first wheel and pinion; R_2, r_2 = the radii of the second; R_3, r_3 = the radii of the third; then

The advantage gained by the screw $= \frac{2\pi a}{d}$,

The advantage gained by the first wheel and pinion $= \frac{R_1}{r_1}$,

and so on.

∴ Total advantage gained or

$$\frac{W}{P} = \frac{2\pi a R_1 R_2 R_3}{d r_1 r_2 r_3}.$$

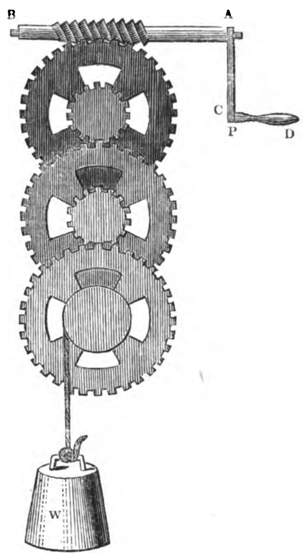


Fig. 86.

Exercises for the Student.

On the Mechanical Powers.

1. A uniform lever l feet long has a weight W lbs. suspended from its extremity; it is required to find the position of the fulcrum, when the long end of the lever balances the short end with the weight attached to it, supposing each unit of length of the lever to be w lbs. *Ans.* $\frac{l^2 w}{2(W+lw)}$ the short arm.

2. A lever, l feet long, is balanced when it is placed upon a prop $\frac{1}{4}$ of its length from the thick end; now when a weight of W lbs. is suspended from the small end the prop must be shifted $\frac{l}{2}$ feet towards it, in order to maintain equilibrium; required the weight of the lever. *Ans.* $\frac{1}{2}$W.

3. A lever, l feet long, is balanced on a prop by a weight of W lbs.; first, when the weight is suspended from the thick end the prop is a feet from it; secondly, when the weight is suspended

from the small end the prop is b feet from it; required the weight of the lever. *Ans.* $\frac{W(a+b)}{l-(a+b)}$ lbs.

4. The forces P and Q act at the arms a and b, respectively, of a straight lever. When P and Q make angles of 30° and 90° with the lever, show that $P=\frac{2bW}{a}$ when equilibrium takes place.

5. Supposing the beam of a false balance to be uniform, a and b the lengths of the arms, P and Q the apparent weights, and W the true weight; then when the weight of the beam is taken into account $\frac{a}{b}=\frac{P-W}{W-Q}$.

6. If a be the length of the short arm, in Example 1., what must be the length of the whole lever, when equilibrium takes place? *Ans.* $a+\sqrt{\frac{2aW}{w}+a^2}$.

7. A beam, l feet long, and weight w lbs., is supported at its extremities by two props, A and B; a weight of W lbs. is suspended at a feet from A, and another weight W_1 at a_1 feet from A; required the pressure on each prop.

Ans. Pressure on $B=\frac{Wa+W_1a_1}{l}+\frac{1}{2}w$, and

" " $A=\frac{W(l-a)+W_1(l-a_1)}{l}+\frac{1}{2}w$.

8. Two given weights, P and Q, hang vertically from two points in the rim of a wheel turning on an axis; find the position of the weights when equilibrium takes place, supposing the angle contained between the radii drawn to the points of suspension to be 90°, and that θ is the angle which the radius, drawn towards P's point of suspension, makes with the vertical. *Ans.* $\tan\theta=\frac{Q}{P}$.

9. What is the ratio between the radii of a wheel and its axle, when a lb. balances a cwt.? *Ans.* 112 to 1.

10. In the combination of wheels and axles represented by *fig.* 128., if a be the advantage gained by the first wheel and axle, and a_1 the advantage gained by the second; show that the whole advantage gained is $a\times a_1$.

11. Show from eq. (1), Art. **127.**, that the principle of work applies to the compound wheel and axle.

12. Demonstrate the principle of work (see Art. **136.**) as applied to some other system of pulleys.

13. In the pulleys represented in *fig.* 76., Art. **135.**, if the cords going round the moveable pulley, form an angle of 120°, then P=W.

14. In the annexed system of two moveable pulleys, the cords at each pulley are inclined 60° to each other; show that W=3P.

Derive the same result on the principle of work. If there are n such moveable pulleys, show that $W=3^{\frac{n}{2}}P$

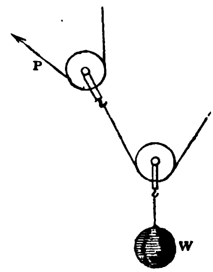


Fig. 87.

15. A power of P lbs., acting parallel to an inclined plane, sustains a weight of 2P lbs.; required the inclination of the plane to the horizon, and the pressure on the plane. *Ans.* Inclination=30°, and pressure =1·733P lbs.

16. What must be the inclination of an inclined plane so that the pressure upon it may be equal to one half the weight?

Ans. 60°.

17. The base of an inclined plane is 8 feet, the height 6 feet, and W=10 cwts.; required P, and the pressure on the plane.

Ans. 6 cwts., and pressure=8 cwts.

18. A weight is supported on an inclined plane by a cord as in *fig.* 78., Art. **140.**; when the inclination of the plane to the horizon is 30°, and the inclination of the cord to the vertical 30°, show that $W=P\sqrt{3}$.

19. In the common vice, if a be the advantage gained by the lever, and a_1 by the screw; show on the principle of the equality of work, that $a \times a_1$ is the total advantage of pressure gained, friction being neglected.

20. Demonstrate the formula given in Art. **150.**, assuming the principle of the equality of work to be true.

21. Two given forces, P and Q, acting at the extremity of a rod moveable freely round its other end, keep it at rest. When the direction of one of the forces is given, show how the direction of the other may be found.

22. A uniform heavy rod, weighing W lbs., is supported in a horizontal position by two equal strings attached to its extremities and to a fixed point, the rod and the strings forming an equilateral triangle. Required the tension of one of the strings. *Ans.* $\frac{W}{\sqrt{3}}$.

23. The directions of two forces, P and Q, which act on a bent lever and keep it at rest, make equal angles with the arms of the

lever, which are a and b feet respectively; required the ratio of P to Q. *Ans.* $\frac{Q}{P}=\frac{a}{b}$.

24. Two weights keep a horizontal lever at rest; the pressure on the fulcrum is 10 lbs., the difference of the weights is 4 lbs., and the difference of the lengths of the arms is 9 inches; required the weights, and the arms of the lever. *Ans.* The weights 3 lbs. and 7 lbs.; and the arms $6\frac{3}{4}$ in. and $15\frac{3}{4}$ in.

25. A weight of 15 lbs. is supported on an inclined plane by a power acting parallel to the plane. The pressure on the plane is 9 lbs.; required the power. *Ans.* 12 lbs.

26. A weight of W lbs. is sustained on an inclined plane by a string parallel to the plane and fixed to the extremity of one of the equal arms of a horizontal lever, to the opposite extremity of which is attached a weight of Q lbs.; required the inclination, α, of the plane. *Ans.* $\sin\alpha=\frac{Q}{W}$.

27. A weight of W lbs. is suspended from the block of a single moveable pulley, and the end of the cord in which the power acts is fastened at the distance of b feet from the fulcrum of a horizontal lever, a feet long, of the second kind; required the force Q which must be applied perpendicularly at the extremity of the lever to sustain W. *Ans.* $Q=\frac{Wb}{2a}$.

CHAP. VII.

APPLICATIONS. — TENSIONS OF CORDS, ETC. THRUST OF BEAMS. PRESSURES OF ROOFS. EQUILIBRIUM OF STRUCTURES. THEORY OF ARCHES. WORK IN MOVING A BODY ON AN INCLINED PLANE. WORK IN RAISING MATERIAL HAVING A GIVEN FORM. WORK IN OVERTURNING BODIES. RELATIVE TO THE ANGLE OF FRICTION.

151. TENSIONS OF CORDS, ETC. — THRUST OF BEAMS, ETC.

1. A platform AB, turning on a hinge at B, is supported from

falling by means of a chain or cord AD, fixed to a hook at D in the same vertical line with B; a weight Q is placed on the platform: it is required to find the tension of the chain, or the force tending to break it.

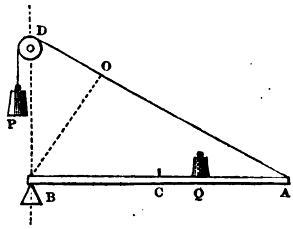


Fig. 88.

Here the platform AB may be regarded as a rod or lever turning on B as a centre of motion. The pressures tending to move the lever *downwards* are the weight Q, and the weight of the lever itself acting in its centre of gravity or middle point C. The pressure tending to move the lever *upwards* is the force P stretching the chain AD, acting with the effective leverage BO, the perpendicular let fall on AD from the centre of moments B. Now as these forces are supposed to balance each other, we have, by the equality of moments, Art. **117.**,

$$\text{Tension chain or } P \times BO = \text{wt. } Q \times QB + \text{wt. platform} \times CB.$$

Let $a = AB$, $q = QB$, $h = BD$, $w =$ the weight of the platform, $P =$ the tension of the chain; $\therefore CB = \frac{1}{2} AB = \frac{1}{2} a$, and by the similar triangles, ABD and AOB, $BO = \frac{AB \times BD}{AD} = \frac{ah}{\sqrt{a^2+h^2}}$; hence by substituting these values, we have

$$P \times \frac{ah}{\sqrt{a^2+h^2}} = Q \cdot q + w \cdot \tfrac{1}{2} a,$$

$$\therefore P = \frac{\sqrt{a^2+h^2}}{2ah}(2Q \cdot q + a \cdot w).$$

Or thus,

Let $\theta = \angle BAD$, then $BO = a \sin \theta$, and in this case the equation of moments becomes

$$P \times a \sin \theta = Q \cdot q + w \cdot \tfrac{1}{2} d,$$

$$\therefore P = \frac{2Qq + aw}{2a \sin \theta}.$$

To find θ when the tension of the chain is given, we have

$$\sin \theta = \frac{2Qq + aw}{2aP}.$$

2. To find the pressure on the hinge in the last problem.

Let C be the common centre of gravity of the platform and the load placed upon it. Put W=the weight of the platform and the load upon it, a=AB, h=BD, c=BC, ∴ AC $=a-c$.

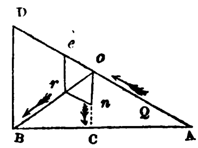


Fig. 89.

Draw Co perpendicular to AB cutting AD in o; then as Co is the direction of the pressure W and oD that of the force stretching the chain, Bo will be the direction of the pressure on the hinge (see Art. 68.). Take on equal to the units in W, and construct the parallelogram of pressures $nreo$, then the units in or will give the amount of the pressure on the hinge.

To find the angle ABo.

From the similar triangles ABD and ACo,

$$\text{C}o=\frac{\text{AC}\times\text{BD}}{\text{AB}}=\frac{(a-c)h}{a},$$

$$\therefore \tan \text{AB}o=\frac{\text{C}o}{\text{BC}}=\frac{(a-c)h}{ac},$$

which gives the direction of the pressure on the hinge.

To find the value of or.

$$\text{B}o=\sqrt{\text{BC}^2+\text{C}o^2}=\frac{1}{a}\sqrt{a^2c^2+(a-c)^2h^2},$$

and from the similar oBD and ore, we have

$$or=\frac{er\times\text{B}o}{\text{BD}}=\frac{\text{W}}{ah}\sqrt{a^2c^2+(a-c)^2h^2},$$

which gives the amount of pressure upon the hinge.

3. A rope AD supports a uniform pole OD, resting on the ground at O, and carrying the weight W, suspended from D; it is required to find the tension of the rope, &c.

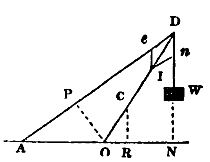


Fig. 90.

Let OP be perpendicular to AD, CR and DN perpendicular to AN;

then taking O as the axis of moments, we have

$$\text{tension rope}\times\text{OP}=\text{W}\times\text{ON}+\text{wt. pole}\times\text{OR}.$$

Put AO$=b$, OD$=a$, $\theta=\angle$NOD, $\alpha=\angle$ODA;

then $OP = a \sin\alpha$, $ON = a \cos\theta$, $OR = \frac{1}{2}ON = \frac{1}{2}a\cos\theta$, and wt. pole $= w$; hence we have, by substituting these values and reducing,

$$\text{tension rope} = \frac{\cos\theta}{\sin\alpha}(W + \tfrac{1}{2}w) \ldots \quad (1).$$

When the rope AD is horizontal, as in *fig.* 91., $\alpha = \theta$, and $\frac{\cos\theta}{\sin\theta} = \cot\theta$, in this case eq. (1) becomes

$$\text{tension rope} = \cot\theta\,(W + \tfrac{1}{2}w) \ldots \quad (2).$$

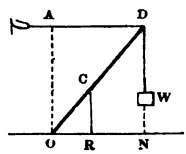


Fig. 91.

This expression increases as θ is decreased, and when θ is very small the tension of the rope becomes very great, which shows the danger there is in breaking the rope when the inclination of the pole to the horizon is small.

To find the thrust upon the pole, when its weight is neglected.

Here the three forces which maintain one another in equilibrium are, the weight W acting in the vertical line DN, the tension of the cord DA, and the thrust on the pole DO. Take Dn equal to the units in W, and construct the parallelogram of pressures DnIe; then the units in De will give the tension of the rope, and the units in DI the thrust upon the pole.

Now from the triangle DnI, we have

$$\frac{DI}{Dn} = \frac{\sin\angle InN}{\sin\angle DIn} = \frac{\cos\angle OAD}{\sin\angle ODA};$$

put $\alpha = \angle ODA$, and $\beta = \angle OAD$, then

$$DI = \frac{W\cos\beta}{\sin\alpha},$$

which gives the thrust on the pole.

When the cord AD is horizontal $\beta = 0$, and then

$$DI = \frac{W}{\sin\alpha}.$$

This expression is a minimum when $\sin\alpha$ is a maximum, that is, when $\alpha = 90°$; and as α is decreased the pressure on the pole is increased, which shows the danger there is in breaking the pole when its inclination to the horizon is small.

4. A beam FB is suspended by two cords, FA and BH; it is required to find the tensions of the cords when the beam is at rest.

Let C be the centre of gravity of the beam; produce the directions of the cords until they intersect in K; join KC; then this line will be vertical, being in the direction of the force of gravity acting on the beam. Take KI equal to the units of weight in the beam, and construct the parallelogram of forces INKV; then the units in KN will give the tension of the cord AF, and the units in KV that of the cord HB.

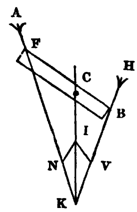


Fig. 92

Put w=KI the wt. of the beam; $\alpha=\angle$AKC, and $\beta=\angle$HKC; then from the triangle IKN, we have

$$\frac{\text{KN}}{\text{KI}}=\frac{\sin\angle\text{KIN}}{\sin\angle\text{INA}};$$

but $\angle$KIN$=\angle$HKC$=\beta$, and $\angle$INA$=\alpha+\beta$,

$$\therefore\ \text{KN}=\frac{w\sin\beta}{\sin(\alpha+\beta)},$$

which gives the tension of the cord AF. Similarly we have for the tension of the cord HB,

$$\text{KV}=\frac{w\sin\alpha}{\sin(\alpha+\beta)}.$$

5. A gate AH is supported by a pin turning in a socket at O, and prevented from falling in the direction AD by a hook and loop at A; it is required to determine the pressure on the pin O, &c.

Let DC be a vertical line passing through the centre of gravity of the gate, and AD the horizontal direction of the force tending to draw the hook out of the wall. Join DO, then this line will be the *direction* of the pressure on the pin (see Art. 68.). Take DC$=w$ the units of weight in the gate, and construct the parallelogram of pressures; then the units in DQ will give the pressure on the pin.

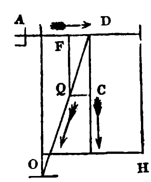


Fig. 93.

Now from the similar triangles DOF and DQC, we have

$$\text{DQ}=\frac{\text{DC}\times\text{OD}}{\text{OF}}.$$

Put h=OF, b=FD, and w=DC the wt. of the gate; then OD$=\sqrt{b^2+h^2}$, and

$$\text{DQ}=\frac{w}{h}\cdot\sqrt{b^2+h^2},$$

which gives the pressure on the hinge.

$$\text{If } \theta = \angle \text{DOF, then } \tan\theta = \frac{b}{h}.$$

152. STABILITY OF STRUCTURE.

1. Let AC be the vertical section of a pillar, having a square base, acted upon by a horizontal pressure P applied at the upper edge, to determine the conditions of equilibrium, supposing the pillar to be overturned on the edge A.

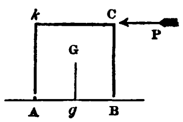


Fig. 94.

Let Gg be a vertical line passing through the centre of gravity G; put a=AB the side of the base, h=BC the height, and w=the weight of each cubic foot of the material; then we have, by the equality of moments,

$$\text{P} \times \text{BC} = \text{wt. pillar} \times \text{A}g,$$

$$\text{or P} \times h = a^2 h w \times \frac{a}{2},$$

$$\therefore \text{P} = \tfrac{1}{2} a^3 w.$$

Now as this expression is independent of h, it follows that all square pillars standing on equal bases require the same force to overturn them, whatever may be their perpendicular heights, provided that force is applied at the top or at the middle of the pillar. Hence the force of the wind, tending to overturn a structure, increases with the height; because the pressure of a given wind, other things being the same, is applied at the middle of the pillar, and its force is proportional to the height of the pillar.

2. To find the point at which the pressure P must be applied, so that the pillar in the last problem may be upon the point of sliding at the same time that it is upon the point of being overturned on the edge A, the coefficient of friction being f.

Let W=the wt. of the pillar, a=the side of the base, and x= the height at which P must be applied; then as P must be equal to the resistance to friction, we have

$$\text{P} = f\text{W};$$

and from the equality of moments

$$\text{P} \times x = \text{W} \times \frac{a}{2},$$

$$\therefore x = \frac{a\text{W}}{2\text{P}} = \frac{a\text{W}}{2f\text{W}} = \frac{a}{2f}.$$

3. Suppose the base of the pillar in Prob. 1., Art. **152.**, to be a rectangle, whose area $=$A, and breadth AB$=a$; then we get

$$P \times h = A h w \times \frac{a}{2},$$

$$\therefore \; P = \tfrac{1}{2} A w a.$$

Here the force necessary to overturn the pillar increases with the breadth of the base when its area is constant.

4. Let ORHF represent a pillar acted upon by a pressure P, in the direction PY, to determine the conditions of stability, &c., supposing the pillar to be overturned upon the edge O.

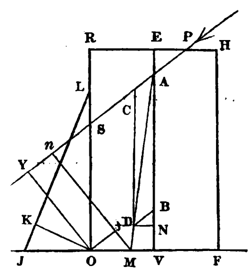


Fig. 95.

Take EV a vertical line, passing through the centre of gravity of the pillar, and cutting the direction of the force P in A; from O let fall OY perpendicular to PY. Put $a=$OV$=$RE, $e=$EP, $h=$OR, $\theta=\angle$EAP, and W$=$the wt. of the pillar; then by the principle of the equality of moments, we have

Momt. P$=$P$\times$OY, and momt. pillar $=$W$\times$OV.

$$\text{But RS} = \text{RP} \times \cot \theta = (a+e) \cot \theta,$$

$$\text{SO} = \text{RO} - \text{RS} = h - (a+e) \cot \theta,$$

$$\text{OY} = \text{SO} \times \sin \theta = \{h - (a+e) \cot \theta\} \sin \theta$$

$$= h \sin \theta - (a+e) \cos \theta$$

$$\therefore \; \text{Momt. P} = \text{P}\{h \sin \theta - (a+e) \cos \theta\},$$

and momt. pillar$=$Wa.

Now when the moment of P is less than the moment of the pillar, it will stand, and *vice versâ*; but if these moments are equal to each other, the pillar will be upon the point of overturning; in this case, we have

$$\text{P} = \frac{\text{W}a}{h \sin \theta - (a+e) \cos \theta} \ldots (1),$$

which gives the value of P when the pillar is upon the point of being overturned on the edge O.

The investigation may also be conducted in the following manner.

Take AB equal to the units in W, and AC equal to the units in P; construct the parallelogram of pressures ABDC, then AD will

be the amount and direction of the resultant tending to overturn the pillar. If AD produced intersect the base within the edge O, the pillar will stand; and on the contrary if the point of intersection, M, fall without the base, the pillar will fall; but if it intersect at the edge O, then the pillar will just be upon the point of overturning.

153. Modulus of Stability.—A structure will be more or less stable, according as the point of resistance, M, is more or less distant from the edge O. Hence the modulus of stability may be defined to be, the ratio that OM bears to OV.

1. To determine the point of resistance M.

Let fall DN perpendicular to EV; then by the similar triangles AVM and AND, we have

$$MV = \frac{AV \times DN}{AN}.$$

$$\text{But } AV = EV - EA = h - e \cot\theta,$$

$$DN = DB \times \sin DBN = P \sin\theta,$$

$$AN = AB + BN = W + P\cos\theta;$$

$$\therefore MV = \frac{(h - e\cot\theta)\,P\sin\theta}{W + P\cos\theta}$$

$$= \frac{(h\sin\theta - e\cos\theta)\,P}{W + P\cos\theta} \ldots (2).$$

Here the stability increases as the value of MV is decreased, but this will take place when the value of e is increased, the other quantities remaining the same; that is to say, the stability will be increased as the point of application of the force, P, is removed from R.

When $MV = OV = a$, the pier is upon the point of falling, and then this expression may be reduced to the form given in eq. (1).

Or thus,

The following method of investigating this problem is highly instructive.

As the point M lies in the line of the resultant, we may take it as the axis of moments (see Art. **68.**). Put $m = OM$, and let P be calculated for one foot of length of the pier, then, putting w for the weight of each foot of the material, we have $W = 2ahw$. Let fall Mn perpendicular to PY, and Ot perpendicular to Mn, then Mn = OY + Mt. But OY has been found in Art. **152.**, and

Mt = OM cos OMt = $m\cos\theta$; hence by substitution we find

$$Mn = h\sin\theta - (a+e)\cos\theta + m\cos\theta$$
$$= h\sin\theta - (a+e-m)\cos\theta;$$

$$\therefore \text{ Momt. } P = P \times Mn$$
$$= P\,\{h\sin\theta - (a+e-m)\cos\theta\} \quad \ldots (3).$$

$$\text{Momt. pier} = \text{wt. pier} \times MV$$
$$= 2ahw \times (a-m).$$

$$\therefore P\,\{h\sin\theta - (a+e-m)\cos\theta\} = 2ahw\,(a-m) \quad \ldots (4).$$

which is the general equation of stability. Eq. (2) may be reduced to the same form.

To find the height of the pier when it has a given stability, $OM = m$.

Solving eq. (4) for the value of h, we find

$$h = \frac{P\cos\theta(a+e-m)}{P\sin\theta - 2aw(a-m)} \quad \ldots (5).$$

When $m=0$, the pier is on the point of being overturned, and then

$$h = \frac{P\cos\theta\,(a+e)}{P\sin\theta - 2a^2w} \quad \ldots (6),$$

which expresses the greatest height of the pier.

2. To find the thrust upon a shore JL when the pier is upon the point of being overturned.

Let $OK = p$ be the perpendicular on the shore; and Q = the thrust upon the shore; then we have by the equality of moments,

$$P \times OY = W \times OV + Q \times OK;$$

$$\therefore Q = \frac{1}{p}(P \times OY - W \times OV),$$

hence we have, by substituting the values of OY and OV given in problem 4., Art. **152**.

$$Q = \frac{1}{p}\left[P\,\{h\sin\theta - (a+e)\cos\theta\} - Wa\right] \quad \ldots (7).$$

3. The stability of a wall ADF supported by buttresses of equal heights and bases.

Conceive the buttresses to be reduced in height and extended

along the wall with the breadth, OD, of their bases unchanged, and let ODBR be the section of the buttress thus formed. Put $DF=c$, $OD=c_1$, $DA=h$, $OR=DB=h_1$, $s=$the perpendicular distance of the point P from OR produced, $w=$the wt. of each cubic foot of the material, $m=OM$ the modulus of stability, and so on to other notation adopted in Art. **153.**; then M being a point in the resultant of the pressures, the sum of the moments, about M as an axis, of the parts of the structure will be equal to the moment of the pressure P.

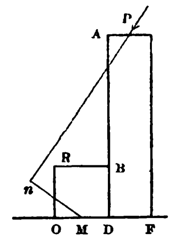


Fig. 96.

The moment of P is given in eq. (3), Art. **153**., where s must be put for $a+e$.

Moment $ADF=$wt. $ADF\times$distance of its centre of gravity from $M=chw\times(c_1-m+\frac{1}{2}c)$.

Moment $ODBR=c_1h_1w\,(\frac{1}{2}c_1-m)$.

$$\therefore\ P\{h\sin\theta-(s-m)\cos\theta\}=chw\,(c_1-m+\tfrac{1}{2}c)+c_1h_1w(\tfrac{1}{2}c_1-m),$$

which expresses the conditions of stability.

By making $m=0$ in this formula, we obtain an expression giving the relation of the elements when the structure is upon the point of being overturned on the outer edge O.

4.. The stability of an embankment whose section has the form of a trapezoid ACRH.

Draw HB parallel to the vertical side RC; let Q be the centre of gravity of the triangle ABH, and G that of the rectangle BR; draw Qq and Gg perpendicular to AC. Put $AC=a$, $HR=b$, $CR=h$, the perpendicular let fall from M on the direction of $P=p$, $w=$the wt. of each cubic foot of the material $P=$the pressure calculated for each foot of length of the embankment, $m=AM$; hence we have

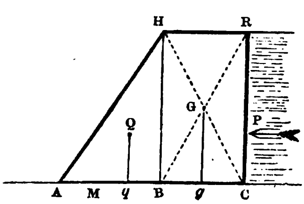


Fig. 97.

$$\text{Momt. } ABH=\text{wt. } ABH\times Mq$$
$$=\tfrac{1}{2}(a-b)\,hw\times\{\tfrac{2}{3}(a-b)-m\};$$
$$\text{Momt. } BR=\text{wt. } BR\times Mg$$
$$=bhw\times(a-m-\tfrac{1}{2}b);$$
$$\text{and Momt. } P=P\,.\,p\,;$$

$\therefore$ P $. p = \frac{1}{2}(a-b)hw\{\frac{2}{3}(a-b)-m\} + bhw(a-m-\frac{1}{2}b) \ldots (1)$,

which gives the conditions of stability.

THE LINE OF RESISTANCE.

154. To determine the equation of the line of resistance, $PAMM_1$, of a rectangular pillar acted upon by a pressure P, as in Art. **152**., Prob. 4.

Let OF be any horizontal joint of the pillar, and M the point of resistance (see Art. **153**.), then the value of MV is given in eq. (2), Art. **153**.

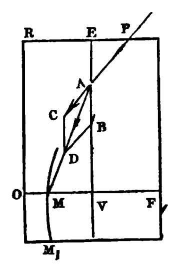


Fig. 98.

Put y for MV, and x for h or EV; that is let us take EV and ER as the axes of coordinates. Moreover, let w=the weight of each unit of surface of the pillar*; then $W=2awx$; substituting these values, we get

$$y = P \cdot \frac{x \sin\theta - e\cos\theta}{2awx + P\cos\theta} \ldots (1)$$

which is the equation of a HYPERBOLA.

If $x=0$, $y=-e=$EP, that is to say, the curve passes through P the point where the pressure is applied. If $y=0$; $x=e.\frac{\cos\theta}{\sin\theta}=$ EA, that is to say, the curve passes through A the point where the direction of the pressure P cuts the vertical through the centre of gravity of the pillar.

If P be applied perpendicularly to the side of the pillar, $\theta=90°$, and then eq. (1) becomes

$$y = \frac{P}{2aw} = \text{a constant},$$

in this case, therefore, the locus is a straight line parallel to EV; and consequently the modulus of stability of the pillar would be the same whatever might be its height.

The mass ROF would slip upon the joint OF, if the angle which the resultant pressure, AM, makes with the vertical, is greater than the angle of friction. In stone and brick structure this is a condition which very rarely takes place.

* If the length of the pillar be 1 foot, then w will represent the weight of each cubic foot.

THEORY OF ARCHES.

155. The highest stone in an arch is called the key-stone, and those which rest upon the abutments or piers are called the springers. An arch is usually included between two curves called the intrados and extrados. Experience has shown that arches almost always give way by certain portions rotating or turning round upon the inner edge of the joints*, and not by slipping, as some authors have assumed in their investigations. We shall therefore only consider the conditions of equilibrium in reference to the former assumption.

LINE OF RESISTANCE IN THE ARCH. POINT OF RUPTURE.

Let ABCD represent any arched structure; P a pressure applied at the joint DC; Ee, Ff, &c., the joints of the arch stones; $R_1 a_1$ the direction of the resultant of the pressure P and the weight of the mass ED acting through its centre of gravity; $R_2 a_2$ that of the pressure P and the weight of the mass FD; $R_3 a_3$ that of the pressure P and the weight of the mass a_3D; and so on: then the curve passing through the points

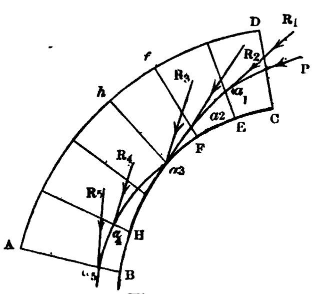


Fig. 99.

P, a_1, a_2, a_3, &c., is the line of resistance. If this curve lies entirely within the face of the arch stones the structure will stand, and on the contrary it will fall if the curve of resistance cuts the intrados or extrados; but if the curve of resistance touches the intrados, as represented at a_3, then the structure will be upon the point of turning on a_3, and this point is called the point of RUPTURE, or the place where the arch first yields under the circumstances of pressure. Hence it appears that the rupture of an arch takes place at the point where the line of resistance TOUCHES the intrados, or, what amounts to the same thing, where the resultant of the pressure P and the weight of the mass, acting through its centre of gravity, TOUCHES the intrados.

* Arches which fall by rotation on the exterior edges may be regarded as exceptional cases.

When the centres of an arch are taken away, the crown almost invariably sinks; this occasions the joint at the crown to open at its lower edge, and at the same time a certain portion, DVPN, of the arch to turn upon D as a centre, thereby producing a rupture, or opening, in the exterior edge at this point. The same effect will take place in the other half of the arch. These two equal portions thus tending to break away from the general mass, exert a horizontal pressure P along the line PC, and thereby cause the walls of the structure to turn on their outer edges. The arch will undergo a rupture at that point where the portion, DVPN, so breaking away, will produce the greatest horizontal thrust; for this point must, obviously, be the yielding part of the arch.

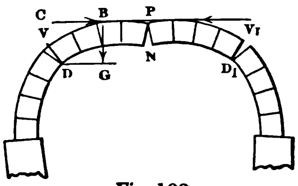


Fig. 100.

Approximate Methods for finding the Point of Rupture in an Arch.

156. If D be the point of rupture; DVPN the portion broken from the semi-arch (see *figs.* 100. and 101.); PC a horizontal line being the direction of the pressure P; DG parallel to PC or perpendicular to PK; BG a vertical line, passing through the centre of gravity of the mass DVPN, and intersecting PC in the point B; then BD will be the direction of the resultant of the pressure P and the weight of the mass DP, and therefore by Art. **155.**, BD will touch the intrados; this property forms the basis of the first method here given for finding the point of rupture.

Again we have by the equality of moments (see *fig.* 100.)

$$P \times BG = \text{wt. mass } DP \times DG,$$

$$\therefore\ P = \frac{\text{wt. mass } DP \times DG}{BG} \ldots (1);$$

now, Art. **155.**, for the point of rupture this value of P is greater than any other value of P derived for any other point in the arch. It is obvious that the expression of eq. (1) really does admit of a maximum value; for whilst the factors of the numerator are increased the denominator is at the same time increased, and so on conversely; so that a point in the arch may be found where this fraction is greater than it is for any other point. This principle is employed in the second method here given for finding the point of rupture.

First Method.

157. Having drawn the arch upon a scale of about an inch to a foot; ASSUME D to be the point of rupture; find DG by eq. (1), Art. **104.**, or more accurately by eq. (4), Art. **106.**; draw the vertical GB, which will be the direction in which the weight of the mass acts, cutting the direction in which P acts in the point B; join BD; then if BD touches the curve in the point D, this will be the point of rupture; but if the line BD does not form a tangent to the curve the point D is not the point of rupture, and some other point must be assumed, higher or lower, according as B is to the right or left of the point where a tangent to D would cut PC, and the whole work must then be gone over again. The following observations enable us to assume D, in the first operation, within a few degrees of the true point of rupture.

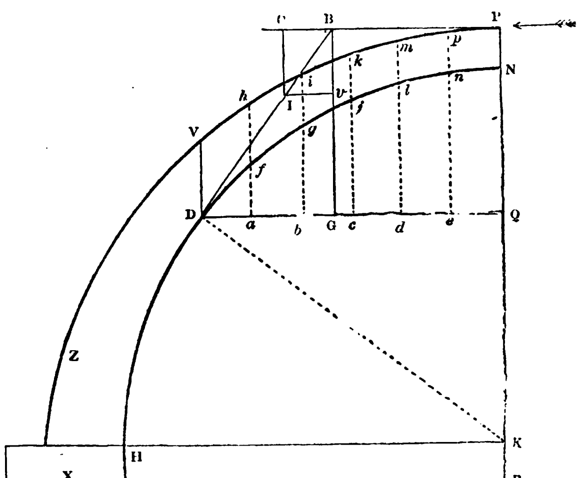


Fig. 101.

In circular arches with parallel extrados, the depth of the arch stones has always a relation to the span of the arch: thus the ratio of the radius of the exterior circle to that of the interior circle will generally lie between 1·5 and 1·1, and the angle of rupture between 66° and 52°; hence the limits of rupture, in such cases, may be taken between these angles. In arches with horizontal extrados, the angle of rupture is never less than 52° and rarely exceeds 66°; so that we may generally assume that the rupture of arches takes place between the angles of 52° and 66°, measured from the crown.

158. *To find the horizontal thrust of the arch.* Take Bv equal to the units of weight in the ruptured portion DVPN, and complete the parallelogram of pressures BvIC; then the units in BC will give the units of pressure in the horizontal thrust P.

If w=the weight of each unit of surface in the arch, then

$$\text{wt. DVPN} = w \times \text{area DVPN}.$$

159. In the following example the line of rupture is assumed to lie in the vertical DV, whereas it more strictly lies in the line Dnt (see *fig.* 55.); but when the depth of the arch stones is comparatively small, the error involved in this assumption is very inconsiderable. However, it should be observed that the centre of gravity of the mass DntRN may be found by eq. (6), Art. **107**.

Example. Let HDN be a semi-circular arch, whose radius KH′ =KN=8 feet; the radius EP of the extrados $9\frac{1}{2}$ feet; and the thickness NP at the crown 9 inches; and the weight of each foot of surface in the arch=200 lbs.; required the point of rupture, and horizontal thrust.

On K as a centre, with the radius KH=8, sweep the semi-arch HDN; mark off NP=·75, the height of the crown; take PE=9·5, and on E as a centre, with the radius EP, sweep the line of the extrados PVZ.

Assume the rupture to take place at D, at the distance of 58° from the crown. Now, in order to ascertain whether or not this is the point of rupture, we proceed as follows:

Draw DQ and PC parallel to HK; divide DQ into six equal parts, viz., Da=ab=bc=cd=&c.; from the points D, a, b, c, &c., draw DV, ah, bi, ck, &c., parallel to QP, cutting the arch in DV, fh, gi, jk, &c.

From the scale we find, DQ=6·73,

$\therefore$ Da, or a of the formula, $=ab=$&c.$=\frac{1}{6}$ of $6{\cdot}73=1{\cdot}12$;

DV or a_0 of the formula$=1{\cdot}6$; PN or $a_6={\cdot}75$; fh or $a_1=1{\cdot}2$; gi or $a_2=1$; jk or $a_3={\cdot}85$; lm or $a_4={\cdot}76$; np or $a_5={\cdot}75$.

Hence we have by the general formula, Art. **104**., eq. (1), for finding the centre of gravity of the arch DVPN;

$$\begin{aligned}\text{DG} &= 1{\cdot}12\{1{\cdot}6+(3\times 6-1){\cdot}75+6(1{\cdot}2+2\times 1+3\times{\cdot}85+4\times{\cdot}76\\ &\qquad +5\times{\cdot}75)\}\div 3\{1{\cdot}6+{\cdot}75+2(1{\cdot}2+1+{\cdot}85+{\cdot}76+{\cdot}75)\}\\ &= 1{\cdot}12\{1{\cdot}6+12{\cdot}75+75{\cdot}84\}\div 3\{1{\cdot}6+{\cdot}75+9{\cdot}12\}\\ &= 101{\cdot}0128\div 34{\cdot}41 = 2{\cdot}9+.\end{aligned}$$

Now mark off DG=2·9, and draw GB parallel to QP, cutting PC in B; join BD, and if this line form a tangent to the curve at D, then this will be the point of rupture.

In the present case BD is very nearly the tangent to the point D, therefore D is very nearly the point of rupture. Hence we have $\angle$ DKN$=58°$ nearly for the distance of the point of rupture from the crown of the arch.

The value of DG might have been determined with greater accuracy by eq. (6), Art. **107.**; and in order to render the work progressive without any erasures, the artifice adopted in Art. **161.** might readily be employed.

For the horizontal thrust we have

$$\text{Wt. DVPN} = 200 \times \text{area DVPN}$$

$$= 200 \times \frac{a}{2}\{a_0 + a_6 + 2(a_1 + a_2 + a_3 + a_4 + a_5)\}$$

$$= 100 \times 1{\cdot}12\{1{\cdot}6 + {\cdot}75 + 2(1{\cdot}2 + 1 + {\cdot}85 + {\cdot}76 + {\cdot}75)\}$$

$$= 1284 \text{ lbs.}$$

From any convenient scale take Bv=1284, and complete the parallelogram of pressures BvIC; then the units in BC=800 nearly, that is, the horizontal thrust of the ruptured portion DVPN will be 800 lbs. nearly.

Second Method.

160. It has been shown that the rupture of an arch takes place at that point where the horizontal thrust (eq. (1), Art. **156.**) is a maximum, that is, where (see *fig.* 100).

$$\frac{\text{wt. mass DP} \times \text{DG}}{\text{BG}}, \text{ or } \frac{\text{area DVPN} \times \text{DG}}{\text{BG}} \text{ is a maximum.}$$

But area DVPN × DG is the moment of the ruptured surface, and BG is the distance of the point of rupture from the crown or horizontal line in which the thrust acts; therefore, *rupture takes place at that point where the moment of the ruptured surface, divided by the distance of the point from the crown, is a maximum.*

In *fig.* 102., let NDm be the intrados; Pnr the extrados of the arch stones; Pa the line of horizontal thrust, intersecting the vertical DV produced in a; Dn and mr joints of the arch stones; nt and rs verticals. Now if D be the point of rupture, the line of rupture will be Dnt, and putting Q for the moment of the ruptured surface DntRN, and h for Da the leverage of P, we have to determine

$$\frac{\text{Q}}{h} \text{ a maximum.}$$

161. In order to apply this principle, so as to give a progressive character to the calculation; assume the points D and m in the intrados, so as to lie without the limits of rupture (see Art. **157.**); draw the vertical lines Da, mk, cutting the line of the horizontal thrust in the points a and k; divide ak into m equal parts in the points b, c, &c.; draw the verticals $b1$, $c2$, &c.; put M' for the moment of the surface DVRN about D, A' its area; M_1' for the moment of 11RN about 1, A_1 for the area 11VD; and so on, M_m' for the moment of mmRN about m, A_m for the area mmVD; $d = ab = bc =$ &c.; $e_0 =$ DV; $e_1 = 11$; $e_2 = 22$; . . . ; $e_m = mm$; $h_0 =$ Da; $h_1 = 1b$; $h_2 = 2c$; . . . ; $h_m = mk$.

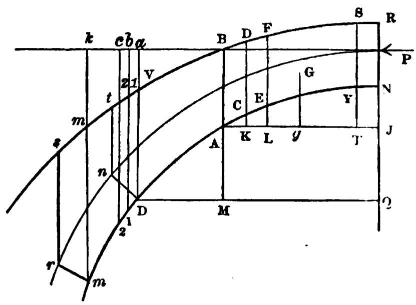


Fig. 102.

Now it will be observed that M' is given in eq. (4). Art. **106.** We now proceed to determine a value for M_{m+1}' in terms of M_m', so that we may be enabled to derive successively the values of M_1', M_2', &c., in such a way that the calculation for any one moment may form a step to the calculation of the one next succeeding.

By eq. (3), Art. **106**, we get

$$M_{m+1}' = M_m' + (A' + A_m)d + \frac{d^2}{6}(e_{m+1} + 2e_m) \dots (1).$$

When $m = 0$, $A_0 = 0$, and $M_0' = M'$,

$$\therefore M_1' = M' + A'd + \frac{d^2}{6}(e_1 + 2e_0) \dots (2).$$

Now $A_m = d(\frac{1}{2}e_0 + e_1 + \dots + e_{m-1} + \frac{1}{2}e_m)$,

therefore by substitution in eq. (1) and reducing, we get

$$M_{m+1}' = M_m' + A'd + d^2\{\tfrac{1}{2}e_0 + e_1 + \dots + e_{m+1} - \tfrac{1}{6}(e_m + 5e_{m+1})\} \dots (3).$$

Let $D_{m+1} = A'd + d^2\{\frac{1}{2}e_0 + e_1 + \dots + e_{m+1} - \frac{1}{6}(e_m + 5e_{m+1})\} \dots (4)$,

$$\therefore M_{m+1}' = M_m' + D_{m+1},$$

$$\therefore M_m' = M_{m-1}' + D_m \dots (5);$$

but $D_{m+1} = D_m + d^2\{e_{m+1} - \frac{1}{6}(e_m + 5e_{m+1}) + \frac{1}{6}(e_{m-1} + 5e_m)\}$

$$= D_m + \frac{d^2}{6}(e_{m+1} + 4e_m + e_{m-1})$$

$$\therefore \text{M}_{m+1}' = \text{M}_m' + \text{D}_m + \frac{d^2}{6}(e_{m+1} + 4e_m + e_{m-1}) \quad . \quad . (6)$$

which exhibits the law by which M_{m+1} is derived from M_m' as expressed in eq. (5).

162. Now by eq. (2), and making $m=1, 2, 3$, &c., successively in eq. (6), we get

$$\begin{aligned} \text{M}_1' &= \text{M}' + \text{A}'d + \frac{d^2}{6}(e_1 + 2e_0) \\ &= \text{M}' + \text{D}_1 \\ \text{M}_2' &= \text{M}_1' + \text{D}_1 + \frac{d^2}{6}(e_2 + 4e_1 + e_0) \\ &= \text{M}_1' + \text{D}_2 \\ \text{M}_3' &= \text{M}_2' + \text{D}_2 + \frac{d^2}{6}(e_3 + 4e_2 + e_1) \\ &= \text{M}_2' + \text{D}_3 \\ \&c. &= \&c. \end{aligned}$$

163. If d be very small, we may suppose, without materially vitiating the result, that, in the expression for M_2', $e_0 = e_2 = e_1$; in the expression for M_3', $e_1 = e_3 = e_2$; and so on; in this case we therefore have

$$\begin{aligned} \text{M}_1' &= \text{M}' + \text{D}_1 \\ \text{M}_2' &= \text{M}_1' + \text{D}_1 + d^2 e_1 \\ &= \text{M}_1' + \text{D}_2 \\ \text{M}_3' &= \text{M}_2' + \text{D}_2 + d^2 e_2 \\ &= \text{M}_2' + \text{D}_3 \\ \&c. &= \&c. \end{aligned}$$

164. Let Q_m = the moment of the whole ruptured surface $\text{N}mrs\text{R}$; $h = mk$ the leverage of P; q_m = the moment of $mrsm$; $s_m = rs$; and p_m = the distance between mm and rs; then we have (see eq. (5), Art. **107**.)

$$\begin{aligned} \text{Q}_m &= \text{M}_m' - q_m \\ &= \text{M}_m' - \frac{p_m^2}{6}(e_m + 2s_m) \quad . \quad . \quad . (7). \end{aligned}$$

Making $m=0, 1, 2$, &c. successively in eq. (7), we get

$$\text{Q} = \text{M}' - \frac{p^2}{6}(e_0 + 2s_0)$$

$$\text{Q}_1 = \text{M}_1' - \frac{p_1^2}{6}(e_1 + 2s_1)$$

$$Q_2 = M_2' - \frac{p_2^2}{6}(e_2 + 2s_2)$$

&c. = &c.

165. Having found convenient expressions for the moments Q, Q_1, Q_2, &c., we proceed next to calculate in succession the values of

$$\frac{Q}{h}, \frac{Q_1}{h_1}, \frac{Q_2}{h_2}, \text{ \&c.},$$

until we arrive at that particular value which is greater than the one which precedes it, as well as that which follows it; then the point on the arch corresponding to this maximum condition gives us the point of rupture.

Let K be put for the maximum value thus found, and let w be put for the weight of each unit of surface in the arch; then

$$\text{the horizontal thrust, } P = \frac{w \times \text{moment surface}}{\text{distance from crown}}$$
$$= w K \ldots (8).$$

To determine the conditions of stability of the Pier of an Arch.

166. Let PND be the semi-arch; OFHR the pier; GV a vertical passing through the centre of gravity of the pier OFHR; gK a vertical passing through the centre of gravity of the semi-arch DHPN; M the point of resistance; put P = the horizontal thrust obtained from eq. (8), Art. **165.**; W = the weight of the pier; W_1 = the weight of the semi-arch DHPN; g = VK; h = PJ the height of the line of horizontal thrust; x = MV; a = OV; m = OM the distance of the point of resistance from the outer edge of the pier. By the equality of moments, taking M as the centre, we get

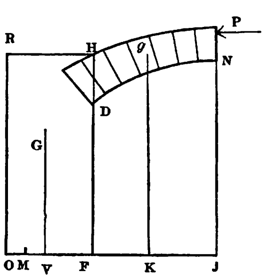


Fig. 103.

$$P \times PJ = W \times MV + W_1 \times MK,$$

$$\therefore\ P \times h = W \times x + W_1(g + x),$$

$$\therefore\ x = \frac{Ph - W_1 g}{W + W_1} \quad \ldots (1),$$

which gives the distance of the point of resistance from the point V. Or, putting $a-m$ for x, we get

$$m=a-\frac{\mathrm{P}h-\mathrm{W}_1 g}{\mathrm{W}+\mathrm{W}_1}\ldots(2).$$

If the section of the pier be rectangular, then $\mathrm{OV}=\mathrm{VF}$, and in this case let $b=\mathrm{OR}$, and $w=$the weight of each unit of section of the pier; then $\mathrm{W}=2abw$, and eq. (2) becomes

$$m=a-\frac{\mathrm{P}h-\mathrm{W}_1 g}{2abw+\mathrm{W}_1}\ldots(3).$$

To determine the breadth of the pier, so as to have a given value of m, we must substitute $a+\mathrm{FK}$ for g in eq. (3), and then solve the resulting quadratic equation for the value of a.

If the pier be upon the point of overturning on its edge O, then $m=0$, and then eq. (3) becomes

$$2a^2bw+a\mathrm{W}_1=\mathrm{P}h-\mathrm{W}_1 g\ldots(4).$$

PRESSURES OF ROOFS.

167. To find the thrust on the rafters of a roof without a tie beam.

Let AC and AB be the rafters of the roof resting upon the side walls C and B. Suppose 2W to be the weight of roof supported by the rafters AC and AB; then, as the weight on each rafter may be conceived to be collected in two equal weights at the extremities, $\frac{1}{2}$W will act vertically upon each wall, and W will act vertically at A. Let $\mathrm{AC}=\mathrm{AB}=l$, $\mathrm{CL}=\mathrm{LB}=b$, and $\mathrm{AL}=e$. Take $\mathrm{AO}=$the units of weight in W, and construct the parallelogram of pressures AEOG, then the units in $\mathrm{AE}=\mathrm{AG}$ will be the thrust on the rafters. Draw ED perpendicular to AL, then $\mathrm{AD}=\mathrm{DO}=\frac{1}{2}\mathrm{W}$, and by the similar triangles CLA and EDA, we have

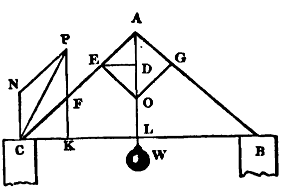


Fig. 104.

$$\mathrm{AE}=\frac{\mathrm{AC}\times\mathrm{AD}}{\mathrm{AL}}=\frac{\mathrm{W}l}{2e}\ldots(1),$$

which gives the thrust upon each rafter.

168. To determine the amount and direction of the pressure of the roof, tending to overturn the side walls.

It has been shown, that besides the thrust upon the rafter just found, there is a vertical pressure upon each wall equal to $\frac{1}{2}$W. Take CN equal to the units in this vertical pressure; also CF equal to AE, the thrust upon the rafter; and construct the parallelogram of pressures CNPF; then CP will give the amount and direction of the pressure of the roof tending to overturn the wall.

Produce PF until it intersects CB in K; then

$$\text{CP}^2=\text{PK}^2+\text{CK}^2 \ldots (1).$$

Now because the triangles CFK and EAD are equal in all respects, $\text{FK}=\text{AD}=\frac{1}{2}\text{W}$; but $\text{PF}=\text{CN}=\frac{1}{2}\text{W}$: therefore $\text{FK}=\text{PF}$; and $\text{PK}=2\text{PF}=\text{W}$. Again, from the similar triangles CFK and CAL, we have

$$\text{CK}=\frac{\text{CL}\times\text{FK}}{\text{AL}}=\frac{b\text{W}}{2e} \ldots (2);$$

substituting the values of PK and CK in eq. (1), and extracting the square root, we find

$$\text{CP}=\sqrt{\text{W}^2+\frac{b^2\text{W}^2}{4e^2}}=\frac{\text{W}}{2e}\sqrt{4e^2+b^2} \ldots (3),$$

which gives the pressure on the wall.*

Let $\alpha=\angle\text{PCN}$, then

$$\cot\alpha=\frac{\text{PK}}{\text{CK}}=\frac{2e}{b} \ldots (4)$$

which gives the inclination of the pressure.

The thrust CF upon the rafter, being resolved into CK and FK, gives 2CK for the force tending to stretch the tie beam CB, therefore, from eq. (2), we find this force to be $\frac{b\text{W}}{e}$.

* To find the inclination of the roof when the pressure CP upon the walls is a minimum, the span remaining the same.

Let w=the weight on each foot of length of the rafter; then $\text{W}=lw$, and eq. (3) becomes

$$\text{CP}=\frac{lw}{2e}\sqrt{4e^2+b^2}=\frac{w\sqrt{e^2+b^2}}{2e}\cdot\sqrt{4e^2+b^2};$$

$$\therefore\ 4e^2+\frac{b^4}{e^2}+5b^2=\text{a minimum};$$

hence we have by differentiating, &c.,

$$\frac{e}{b}=\frac{1}{\sqrt{2}},\ \text{that is, } \tan\text{ACB}=\frac{1}{\sqrt{2}}=\tan 35^\circ\ 16';$$

WORK IN RAISING MATERIAL HAVING A GIVEN FORM.

169. It has been shown, Art. **110.**, that the work in raising any structure, or material, having a given form, is equal to the weight of the whole material in lbs. multiplied by the space in feet through which the centre of gravity is raised.

1. To find the work in raising the material in a rectangular wall.

Let a=the length, b=the thickness, c=the height, w=the weight of a cubic foot of the material, and U=the work; then

$$\text{Wt. wall}=abcw,$$

space through which the centre of gravity of the structure is raised$=\frac{1}{2}c$,

$$\therefore \text{ Work or U}=abcw\times\tfrac{1}{2}c=\tfrac{1}{2}abc^2w.$$

2. To find the work in raising the material of the shaft of a pit.

Let d=the diameter of the pit, a=its depth, and w=the wt. of a cubic foot of the material; then

$$\text{Wt. of the material}=\tfrac{1}{4}\pi d^2aw,$$

$$\therefore \text{ Work or U}=\tfrac{1}{4}\pi d^2aw\times\frac{a}{2}=\tfrac{1}{8}\pi a^2d^2w.$$

3. To find the work in raising a rope from a pit or well.

Let a=the length of rope, c=its circumference, w=the weight of 1 foot length of rope 1 inch in circumference.

Now since the areas of circles are to each other as the squares of their like dimensions,

$$\therefore \text{ Wt. of 1 ft. of rope}=c^2w,$$

$$\therefore \text{ Wt. whole rope}=c^2w\times a,$$

$$\therefore \text{ Work in raising the rope}=c^2wa\times\frac{a}{2},$$

$$=\tfrac{1}{2}a^2c^2w.$$

Required the work, U, when W lbs. of coals are raised by the rope. Here we have

$$\text{U}=\text{work in raising rope}+\text{work in raising coals},$$

$$=\tfrac{1}{2}a^2c^2w+\text{W}a=a(\tfrac{1}{2}ac^2w+\text{W}).$$

4. To find the work in raising water into a cistern from a well.

Let ABCD be a cylindrical cistern, HNR the level of the water in the well, which we shall first consider to remain unchanged, G the centre of gravity of the water in the cistern.

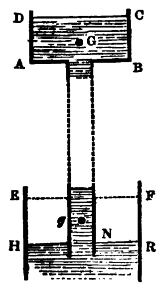


Fig. 104*.

Let a=AD, the depth of the water in the cistern, d=its diameter, e=RB, the distance between the bottom of the cistern and the level of the water in the well.

$$\text{Wt. water raised}=62{\cdot}5\times\tfrac{1}{4}\pi d^2 a,$$

$$\text{NG}=e+\tfrac{1}{2}a,$$

$$\therefore\ \text{Work}=\text{wt. water}\times\text{NG}$$

$$=7{\cdot}8125\pi a d^2(2e+a).$$

Now let us suppose EF to be the level of the water at first in the well, HR the level when the cistern is filled, and that no water flows into the well during the process of pumping. Let g be the centre of gravity of the water HRFE. Put d_1 for the diameter of the well; then

$$\text{Volume water raised}=\tfrac{1}{4}\pi a d^2,$$

$$\therefore\ \tfrac{1}{4}\pi d_1^2\times\text{HE}=\tfrac{1}{4}\pi a d^2,$$

$$\therefore\ \text{HE}=\frac{a d^2}{d_1^2},$$

$$\therefore\ \text{G}g=\text{RB}-\tfrac{1}{2}\text{HE}+\tfrac{1}{2}\text{AD}$$

$$=e-\frac{a d^2}{2 d_1^2}+\tfrac{1}{2}a,$$

$$\therefore\ \text{Work}=\text{wt. water}\times\text{G}g$$

$$=\frac{62{\cdot}5}{4}\pi a d^2\left\{e-\frac{a d^2}{2 d_1^2}+\tfrac{1}{2}a\right\}.$$

If $d_1=d$, then this expression becomes,

$$\text{Work}=\frac{62{\cdot}5}{4}\pi a d^2 e.$$

WORK IN MOVING A BODY ON AN INCLINED PLANE.

170. To determine the work in moving a body up an inclined plane by a pressure acting parallel to the plane, the coefficient of friction being f.

By eq. (1), Art. **139.**,

$$\text{Pressure on the plane} = \text{W} \cdot \frac{\text{AB}}{\text{AC}},$$

$$\therefore \text{ Resistance of friction} = f \cdot \text{W} \cdot \frac{\text{AB}}{\text{AC}},$$

$$\therefore \text{ Work due to friction} = f \cdot \text{W} \cdot \frac{\text{AB}}{\text{AC}} \times \text{AC} = f \cdot \text{W} \cdot \text{AB}.$$

$$\text{Work due to gravity} = \text{W} \times \text{BC},$$

$$\therefore \text{ Total work} = f \cdot \text{W} \cdot \text{AB} + \text{W} \cdot \text{BC} \ \ldots \ (1).$$

If the body descend the plane, then we obviously have

$$\text{Total work} = f \cdot \text{W} \cdot \text{AB} - \text{W} \cdot \text{BC} \ \ldots \ (2).$$

Now expression (1) is independent of the length of the plane, it being, in fact, the work expended in moving the body over the horizontal distance AB, added to the work due to gravity in elevating the body through the vertical space BC. As a curved surface may be regarded as being made up of an infinite number of straight planes, therefore the work upon the whole curve will be equal to the work done upon the horizontal projection of the curve, added to the work done in opposition to gravity. The total work is, obviously, independent of the nature of the curve. *Thus the work done in moving a body from* A *to* C *on the irregular curve* ADC *is equal to the work of friction done upon the horizontal plane* AB *added to the work due to gravity in moving the body through the vertical line* BC.*

Or we have from eq. (1),

Work in moving a body up the curved surface ADC,

$$= \text{W}(\text{BC} + f \cdot \text{AB}) = \text{W}(\text{BC} + \tan\theta \cdot \text{AB}).$$

In order to give a geometrical interpretation to this result, draw CH parallel to AB, and AH to BC; also draw CK making the angle HCK equal to θ, the angle of friction; then $\text{HK} = \tan\theta \cdot \text{AB}$, and therefore we get

Work in moving a body up the curved surface, $\text{ADC} = \text{W}(\text{BC} + \text{HK}) = \text{W} \times \text{AK}$,

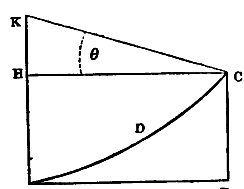


Fig. 105.

That is to say, THE WORK IN MOVING A BODY UP ANY CURVED SURFACE, ADC, IS EQUAL TO THE WORK IN RAISING THE BODY THROUGH THE VERTICAL HEIGHT AK, IN OPPOSITION TO GRAVITY.

* See the Author's "Exercises on Mechanics," &c., p. 130.

171. When the inclination of the plane is small, AC is very nearly equal to AB, and therefore in eq. (1) or (2) we take the length of the plane, AC, in the place of its base AB. The problems given at page 19., Art. **11.**, of the Author's "Exercises on Mechanics," &c., are solved upon this principle.

172. If the body be upon the point of descending by its own weight, the work requisite to move the body down the plane must be equal to nothing, hence eq. (2) becomes,

$$f \cdot W \cdot AB - W \cdot BC = 0,$$

$$\therefore f = \frac{BC}{AB} = \tan \alpha,$$

that is, the inclination of the plane will be equal to the limiting angle of resistance, see Art. **88.**

173. Let P = the pressure acting parallel to the plane, then we have,

Work in moving the body through the length of the plane $= P \times AC$.

Hence we have from eqs. (1) and (2),

$$P \times AC = f \cdot W \cdot AB \pm W \cdot BC,$$

$$\therefore P = f \cdot W \cdot \frac{AB}{AC} \pm W \cdot \frac{BC}{AC},$$

$$= f \cdot W \cdot \cos \alpha \pm W \cdot \sin \alpha,$$

$$= W\{f \cdot \cos \alpha \pm \sin \alpha\} \ldots (1),$$

which is the value of P in order to move the body up or down the plane, as the case may be.

In order to eliminate f, we have by Art. **88.**,

$$f = \tan \theta = \frac{\sin \theta}{\cos \theta},$$

where θ is put for the angle of friction.

Substituting in eq. (1), we have

$$P = W\left\{\frac{\sin \theta \cos \alpha \pm \cos \theta \sin \alpha}{\cos \theta}\right\},$$

$$= W \cdot \frac{\sin (\theta \pm \alpha)}{\cos \theta} \ldots (2).$$

174. Prob. To find the inclination of an inclined plane, when the traction up the plane is P, and the traction down the plane is p.

Here by eq. (1), Art. **173.**, we have

$$P = W\{f \cdot \cos \alpha + \sin \alpha\},$$

$$p = W\{f \cdot \cos \alpha - \sin \alpha\},$$

hence, we have, by addition,

$$P+p=2f W \cos \alpha,$$

$$\therefore \cos \alpha=\frac{P+p}{2f W},$$

which gives the inclination of the plane as required.

WORK IN OVERTURNING BODIES.

175. To find the work expended in overturning a right cone, ABV, see *fig.* 59., page 108.

Put $R=AD$, the radius of the base;

$h=DV$, the perpendicular height;

$w=$no. lbs. weight of each cubic foot of the material;

and $U=$the work necessary to overturn the body;

then we have by eq. (1), Art. **111**.,

$$U=W(\sqrt{x^2+y^2}-y),$$

where $W=$wt. of the cone$=\frac{1}{3}\pi R^2 hw$; $x=AD=R$; and $y=Dg=\frac{1}{4}h$, (see Art. **95**.);

$$\therefore U=\tfrac{1}{3}\pi R^2 hw\{\sqrt{R^2+\tfrac{1}{16}h^2}-\tfrac{1}{4}h\}.$$

If the height of the cone be equal to its diameter, that is, if $h=2R$, then this expression becomes

$$U=\tfrac{1}{3}\pi R^4 w(\sqrt{5}-1).$$

176. To find the work in overturning a right cylinder, the radius of the base being r, the perpendicular height h, and the weight of each cubic foot of the material w.

In this case, $W=\pi r^2 hw$, $x=r$, and $y=\frac{1}{2}h$,

$$\therefore U=\pi r^2 hw(\sqrt{r^2+\tfrac{1}{4}h^2}-\tfrac{1}{2}h).$$

When h is very small as compared with r, then we have very nearly

$$U=\pi r^3 hw=\text{weight}\times r,$$

which expresses the work in raising a thin board from a horizontal position to a vertical one, r being the distance of the centre of gravity from the edge upon which the board is turned.

177. To show that in similar solids the work necessary to overturn them varies as the fourth power of the ratio of their linear dimensions.

Let r be the linear ratio of two similar solids, U and U_1 being the work done upon each; then we have from the general equation,

$$U = W(\sqrt{x^2+y^2}-y),$$

and by substituting rx for x, ry for y, r^3W for W,

$$\begin{aligned} U_1 &= r^3W(\sqrt{r^2x^2+r^2y^2}-ry) \\ &= r^4W(\sqrt{x^2+y^2}-y) \\ &= r^4U. \end{aligned}$$

RELATIVE TO THE ANGLE OF FRICTION.

178. Let AB be a pole resting upon the ground at A, and supported by the cord, BC, acting parallel to the ground; it is required to determine the position of the pole when it is upon the point of slipping.

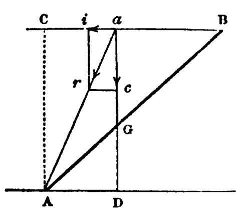


Fig. 106.

Suppose AB to be the position of the pole when it is about to slip; G its centre of gravity; DGa a vertical line passing through G, and intersecting the cord BC in a; AC a vertical line; then aA will be the direction of the pressure on the ground, and as the pole is about to slip, the angle aAC will be equal to the angle of friction.

Put W = the weight of the pole; P = the tension of the cord BC; θ = the angle of friction aAC; $\alpha = \angle$BAC, the inclination of the pole to the vertical; AG $= \frac{1}{n}$ of AB.

From the parallelogram of pressures $acri$, we have

$$\frac{cr}{ca} = \tan \angle car,$$

but $cr = ai =$ P, $ca =$ W, and $\angle car = \angle a$AC $= \theta$,

$$\therefore \frac{P}{W} = \tan \theta.$$

In order to determine another expression for the relation of P to W, conceive the pole to be a lever turning upon A as a centre, then we have by the equality of moments

$$P \times AC = W \times AD;$$

but $AC = AB \cos\alpha$, and $AD = AG \sin\alpha = \frac{1}{n} AB \sin\alpha$;

$$\therefore \; P \times AB \cos\alpha = W \times \frac{1}{n} AB \sin\alpha;$$

$$\therefore \; \frac{P}{W} = \frac{1}{n} \cdot \frac{\sin\alpha}{\cos\alpha} = \frac{1}{n} \tan\alpha.$$

We have now by equality,

$$\frac{1}{n} \tan\alpha = \tan\theta;$$

$$\therefore \; \tan\alpha = n \,.\, \tan\theta,$$

which determines the position of the pole.

$$\text{But } \tan\theta = \frac{Ca}{AC} \text{ and } \tan\alpha = \frac{CB}{AC};$$

$$\therefore \; \frac{CB}{AC} = n \,.\, \frac{Ca}{AC};$$

$$\therefore \; CB = n \,.\, Ca.$$

From this equality we may readily derive an easy geometrical construction.

179. In order to derive a more general formula, let β=the angle ABC, which the cord makes with the pole.

From the parallelogram of pressures (see last figure), we have

$$\frac{ai}{ri} = \frac{\sin \angle ari}{\sin \angle rai};$$

but $ai = P$, $ri = ca = W$, $\angle ari = \angle aAC = \theta$, and $\angle rai = \angle ABC + \angle BAa = \beta + \alpha - \theta$,

$$\therefore \; \frac{P}{W} = \frac{\sin\theta}{\sin(\beta+\alpha-\theta)}.$$

Again we have by the equality of moments, Art. **117.**,

$$P \times AB \sin\beta = W \times AG \sin\alpha;$$

$$\therefore \; \frac{P}{W} = \frac{AG}{AB} \cdot \frac{\sin\alpha}{\sin\beta}$$

$$= \frac{1}{n} \cdot \frac{\sin\alpha}{\sin\beta}.$$

Hence we have by equality,

$$\frac{1}{n} \cdot \frac{\sin\alpha}{\sin\beta} = \frac{\sin\theta}{\sin(\beta+\alpha-\theta)} \; \ldots (1).$$

From this equation α may be determined.

When $\beta=90-\alpha$, this equality becomes the same as that given in the preceding problem.

180. A given pole, AB, rests with its foot on the horizontal plane AK, and with its upper extremity B against an inclined wall KB; it is required to find its position when bordering on a state of motion, having given the place of its centre of gravity G, and the coefficients of friction of the two surfaces.

Let AB be the position of the pole when in a state bordering on motion; $\angle a\text{AC}=\theta$, the angle of friction for the surface AK; $\angle a\text{BE}=\theta_1$, the angle of friction for the surface KB; $\angle\text{KAB}=\alpha$, the inclination of the pole; $\angle\text{AKB}=\phi$, the inclination of the two surfaces to each other; and $\text{AG}=\frac{1}{n}$ of AB.

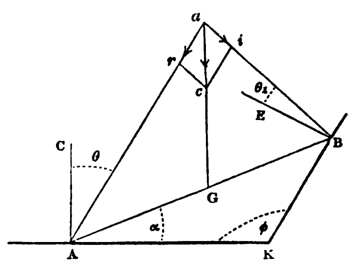


Fig. 107.

Now, as the pole is about to slip at A as well as at B, the forces in equilibrium are,—the weight of the pole, W, acting in the vertical line Ga, the resistance of the ground, acting in the line Aa, and the resistance of the wall, R, acting in the line Ba. Hence we have from the parallelogram of pressures $aicr$,

$$\frac{cr}{ac}=\frac{\sin\angle car}{\sin\angle cra};$$

but $ac=\text{W}$, $cr=ai=\text{R}$, $\angle car=\angle a\text{AC}=\theta$,

$$\begin{aligned}\angle cra&=\angle \text{AB}a+\angle\text{BA}a\\&=\overline{\alpha+\phi-90+\theta_1}+\overline{90-\theta-\alpha}\\&=\phi+\theta_1-\theta;\end{aligned}$$

hence we have by substitution,

$$\frac{\text{R}}{\text{W}}=\frac{\sin\theta}{\sin(\phi+\theta_1-\theta)}.$$

Taking A as the centre of motion, we now have by the equality of moments,

$$\text{R}\times\text{AB}\sin\angle\text{AB}a=\text{W}\times\text{AG}\cos\angle\text{BAK};$$

$$\therefore\ \frac{\text{R}}{\text{W}}=\frac{1}{n}\cdot\frac{\cos\alpha}{\sin(\alpha+\phi+\theta_1-90)}=\frac{\cos\alpha}{n\cos(\alpha+\phi+\theta_1)}.$$

Hence we have by equality,

$$\frac{\cos \alpha}{n \cos(\alpha+\phi+\theta_1)}=\frac{\sin\theta}{\sin(\phi+\theta_1-\theta)} \ldots (1)$$

From which equation the value of α may be determined as required.

When the wall BK is vertical, $\phi=90$, and then (eq. 1) becomes

$$\frac{\cos \alpha}{n \sin(\alpha+\theta_1)}=\frac{\sin\theta}{\cos(\theta_1-\theta)}.$$

$$\text{Now } \frac{\cos\alpha}{\sin(\alpha+\theta_1)}=\frac{\cos\alpha}{\sin\alpha\cos\theta_1+\cos\alpha\sin\theta_1}$$

$$=\frac{1}{\tan\alpha\cos\theta_1+\sin\theta_1};$$

$$\therefore \frac{1}{\tan\alpha\cos\theta_1+\sin\theta_1}=\frac{n\sin\theta}{\cos(\theta_1-\theta)};$$

$$\therefore \tan\alpha=\frac{\cos(\theta_1-\theta)}{n\sin\theta\cos\theta_1}-\frac{\sin\theta_1}{\cos\theta_1}$$

$$=\frac{1}{n}\left\{\frac{1}{\tan\theta}+\tan\theta_1\right\}-\tan\theta_1$$

$$=\frac{1-(n-1)\tan\theta\tan\theta_1}{n\tan\theta} \ldots (2),$$

which gives the position of the pole as required.

If $f=\tan\theta$, the coefficient of friction of the ground; and $f_1=\tan\theta_1$, the coefficient of friction of the wall; then eq. (2) becomes

$$\tan\alpha=\frac{1-(n-1)ff_1}{nf} \ldots (3).$$

To find the relation of P *to* W, *when the body is upon the point of being moved up or down an inclined plane.*

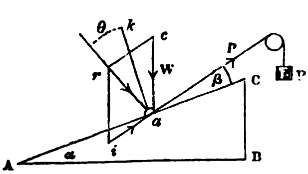


Fig. 108.

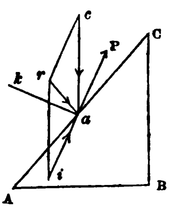


Fig. 109.

181. Let a be the body on the inclined plane ABC; iaP the

direction of the pressure P; ac a vertical line in the direction of the pressure of the weight W; ar the direction of the resultant of these two pressures when the body is in a state bordering upon motion; then the body will be upon the point of moving up the plane or down the plane according as this resultant lies to the left or to the right of the line ak drawn perpendicular to the plane.

Put $\alpha = \angle BAC$, the inclination of the plane;

$\beta = \angle CaP$, the direction of P with respect to the plane;

$\theta = \angle rak$, the angle of friction.

Let ac=the units in W; ai=the units in P; and $acri$ the parallelogram of pressures.

Hence we have from the triangle rac,

$$\frac{rc}{ac} = \frac{\sin \angle rac}{\sin \angle arc};$$

but $rc = ai = P$, and $ac = W$;

$$\therefore \; \frac{P}{W} = \frac{\sin \angle rac}{\sin \angle arc} \ldots (1).$$

Now in *fig.* 108.,

$$\angle rac = \angle rak + \angle kac = \theta + \alpha,$$

$$\angle arc = \angle rai = \angle kai - \angle rak$$
$$= 90 + \beta - \theta.$$

Substituting in eq. (1), we find

$$\frac{P}{W} = \frac{\sin(\theta + \alpha)}{\sin(90 + \beta - \theta)} = \frac{\sin(\alpha + \theta)}{\cos(\beta - \theta)} \ldots (2),$$

which is the required relation when the body is about to move up the plane.

Now, in *fig.* 109.,

$$\angle rac = \angle kac - \angle rak = \alpha - \theta,$$

$$\angle arc = \angle rai = 90 + \beta + \theta.$$

Substituting in eq. (1), we find

$$\frac{P}{W} = \frac{\sin(\alpha - \theta)}{\sin(90 + \beta + \theta)} = \frac{\sin(\alpha - \theta)}{\cos(\beta + \theta)} \ldots (3),$$

which is the required relation when the body is about to move down the plane.

Hence we have generally,

$$\frac{P}{W} = \frac{\sin(\alpha \pm \theta)}{\cos(\beta \mp \theta)} \ldots (4),$$

where the upper sign is to be taken when the body is upon the point of moving up the plane, and the lower sign when it is upon the point of moving down the plane.

No motion can take place so long as the relation of P to W lies between these two values.

Or thus.

Let *ac* represent the weight of the body in direction and magnitude, and *av* that of the pressure P. Draw *ak* perpendicular to the plane, and complete the parallelogram of pressures shown in the figure; then, by resolving the pressures, P and W, parallel and perpendicular to the plane, we get, as in Art. **140**,

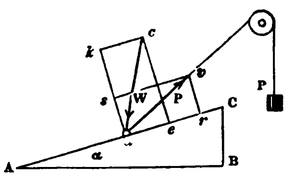


Fig. 110.

∴ Perpendicular pressure on the plane $= ak - as$

$$= W \cos \alpha - P \sin \beta.$$

∴ Resistance of friction = perpendicular pressure × coef. friction

$$= f(W \cos \alpha - P \sin \beta).$$

Pressure tending to move the body up or down the plane

$$= \pm(ar - ae) = \pm(P \cos \beta - W \sin \alpha),$$

where the upper sign is taken when the body is about to move up the plane, and the lower sign when it is about to move down the plane.

Now, when the body is upon the point of moving, the pressure tending to give motion must be equal to the resistance of friction.

$$\therefore P \cos \beta - W \sin \alpha = \pm f \{W \cos \alpha - P \sin \beta\};$$

$$\therefore \frac{P}{W} = \frac{\sin \alpha \pm f \cos \alpha}{\cos \beta \pm f \sin \beta} \ldots (5).$$

where the signs are to be taken as before explained.

By substituting $\tan \theta$ for f, this expression may be reduced to the same form as given in eq. (4).

If the direction of the force P be parallel to the plane, $\beta = 0$, and then the relation given in eq. (5) becomes the same as that of eq. (1), Art. **173.**,

When the plane is horizontal, as in *fig.* 111, $\alpha = 0$, and then eq. (4) becomes

$$\frac{P}{W} = \frac{\sin \theta}{\cos(\beta - \theta)} \ldots (6).$$

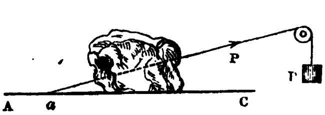


Fig. 111.

When the plane is perfectly smooth, the coefficient of friction, as well as the angle of friction, is nothing; that is, $\theta=0$; and in this case eq. (4) becomes the same as eq. (2), Art. **140.**

182. Let the pressure P be applied so as to *push* the body up the plane, as represented in *fig.* 112., where the same letters of reference and symbols are used as in *fig.* 108., β in this case being put for the angle PaC of *fig.* 112.

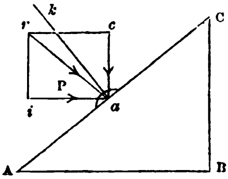


Fig. 112.

From the triangle ari, we have

$$\frac{ai}{ri}=\frac{\sin\angle ari}{\sin\angle rai}=\frac{\sin(\alpha+\theta)}{\sin(\beta-\theta-90)};$$

$$\therefore\ \frac{P}{W}=\frac{\sin(\alpha+\theta)}{-\cos(\beta-\theta)}\ldots(7),$$

which is essentially the same expression as eq. (2), where there is a *pulling* pressure; for it will be observed that $\beta-\theta$ is greater than 90°, and its cosine is consequently negative.

If P acts horizontally, then $\beta=\angle$ PaC$=180-\alpha$, and $-\cos(\beta-\theta)$ $=-\cos(180-\overline{\alpha+\theta})=\cos(\alpha+\theta)$; in this, therefore, eq. (7)

$$\frac{P}{W}=\frac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)}=\tan(\alpha+\theta)\ldots(8).$$

If the plane be perfectly smooth, then $\theta=0$, and eq. (8) becomes the same as eq. (2) Art. **141.**

Least traction on the plane.

183. From eq. (2), Art. **181.**, we have

$$P=W\cdot\frac{\sin(\alpha+\theta)}{\cos(\beta-\theta)},$$

which is an expression for the traction requisite for drawing the weight W up the inclined plane ABC (see *fig.* 108.), the direction of the traction forming any angle β with the direction of the plane. Now, as α and θ are supposed to be constant in the present investigation, this expression will be a minimum when $\cos(\beta-\theta)$ is a maximum; but this will obviously be a maximum when $\beta-\theta=0$, or when $\beta=\theta$: that is to say, the force of traction will be a minimum when the angle of traction is equal to the angle of friction.

In the case of least traction, therefore, we have the following relation between P and W:

$$P = W \sin(\alpha+\theta) \ldots (9).$$

And when the body is moved on the horizontal plane, as in *fig.* 111., we have

$$P = W \sin\theta \ldots (10);$$

where it is to be observed that θ is the angle of traction, as well as the angle of friction.

To determine the work requisite to move a body up an inclined plane when the moving pressure P *is applied at any given angle.*

184. Let AD be the direction of the pressure P, tending to move the body up the inclined plane ABC. Draw CD perpendicular to AD; and put U_1 =the work in moving the body up the plane AC; then, by Art. **70.**,

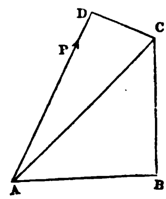


Fig. 113.

$$U_1 = P \times AD = P \times AC \cos\beta;$$

but by eq. (5), Art. **181.**,

$$P = \frac{1}{\cos\beta + f\sin\beta} \cdot (W \sin\alpha + f W \cos\alpha);$$

$$\therefore U_1 = \frac{\cos\beta}{\cos\beta + f\sin\beta} \cdot (W \times AC \sin\alpha + f W \times AC \cos\alpha)$$

$$= \frac{1}{1 + f\tan\beta} \cdot (W \times CB + f W \times AB) \ldots (1).$$

Now, the quantity within the brackets is equal to the work in moving the body up the plane by a pressure acting parallel to the plane (see eq. (1), Art. **170.**). Putting Q_1=the work due to friction done on AB, and q_1=the work due to gravity through BC, the preceding equality becomes

$$U_1 = \frac{1}{1 + f\tan\beta}(Q_1 + q_1) \ldots (2).$$

Now, this expression is independent of the inclination of the plane; hence it follows, reasoning in the same manner as in Art. **170.**, that *the work done in moving a body from* A *to* C (see *fig.* 105.) *on the irregular curve* ADC, *by a pressure having a constant inclination to the plane, is equal to the work of friction done upon the horizontal plane* AB, *added to the work due to gravity in moving the body through the vertical line* BC, *multiplied by the constant* $\frac{1}{1+f\tan\beta}$.

185. The value of U_1 in eq. (1) is maximum when $\tan\beta$ is a minimum, or when $\beta=0$; that is to say, the work is a maximum when the direction of traction is parallel to the plane. Moreover, as β is increased, limited, of course, by the value $\beta=90-\alpha$, the work is decreased. Hence it follows that *the minimum traction does not necessarily give the minimum work.*

When $\beta=\theta$, we have for *the work of the minimum traction,*

$$U_1=\frac{1}{1+f\tan\theta}(Q_1+q_1)$$
$$=\frac{1}{1+\tan^2\theta}\cdot(Q_1+q_1)$$
$$=\cos^2\theta\cdot(Q_1+q_1)\ldots(3).$$

186. To find the angle of traction, β, when the work up an incline is given.

From eq. (2), Art. **184.**, we find

$$\tan\beta=\frac{Q_1+q_1-U_1}{fU_1}\ldots(4).$$

187. *Prob.* A uniform beam, AB, rests on a given cylinder; to find the weight, W, which must be suspended from the extremity A, so that the beam may just be upon the point of sliding off the cylinder.

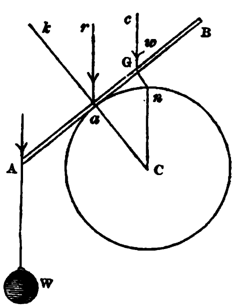


Fig. 114.

Let C be the centre of the cylinder; G the centre of gravity of the beam; a the point of contact when it is about to slip; n the highest point of the cylinder, or the point with which G was in contact before the weight was suspended.

Put w=the weight of the beam; b=AG $=\frac{1}{2}$AB; r=Ca, the radius of the cylinder; θ=the angle of friction. Draw the vertical lines ar, Gc, &c., and produce Ca to k; then ar will be the direction of the resultant of the two parallel pressures, W and w, and the angle rak will be the angle of friction when the beam is about to slip. Now, because ar and Cn are parallel, we have

$$\angle aCn=\angle rak=\theta.$$

Taking the moments about a as a centre, we have

$$W\times Aa=w\times Ga,$$
$$\therefore\ W(AG-Ga)=w\times Ga;$$

$$\text{but AG}=b, \text{ and G}a=\text{arc } an=r\theta,$$

$$\therefore \text{ W}(b-r\theta)=w\times r\theta,$$

$$\therefore \text{ W}=\frac{w\times r\theta}{b-r\theta};$$

which is the expression required.

188. *Prob.* A given cylinder, ABD, whose weight is W, rests between two inclined planes AM and BM; a weight P is suspended by a cord PD coiling round the cylinder; it is required to determine P, so that the cylinder may be upon the revolving.

Let α and β be the inclination of the planes AM and BM respectively; r=OA=CB=CD the radius of the cylinder; $\theta=\angle$ CAO $=\angle$ CBO, the angle of friction.

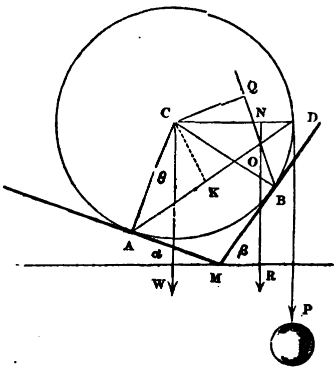


Fig. 115.

Here the body is acted upon by the two vertical pressures W and P; these pressures must produce a vertical pressure NR equal to the sum of these, passing through the point O; for the resultant in the direction NR, when resolved in the directions OA and OB, must just cause the cylinder to slip at A and B at the same moment.

By the equality of moments, Art. **80**.,

$$\text{P}\times\text{DC}=\text{R}\times\text{NC, or}$$

$$\text{P}\times r=(\text{P}+\text{W})\times\text{CO sin CON}.$$

Here we must substitute known values for CO and sin CON.

Let fall CK and CQ perpendiculars on AO and BO produced; then the triangles COK and COQ are identical, and $\angle$ COA= $\angle$ COQ.

Because the sum of the angles OAM and OBM is equal to two right angles, $\angle$ AOB$=\alpha+\beta$.

Hence we have

$$\angle\text{AOB}+2\angle\text{COQ}=180^\circ,$$

$$\therefore \angle\text{COQ or angle COA}=90^\circ-\frac{\alpha+\beta}{2}$$

$$\angle\text{NOQ}=\angle\text{ROB}=\beta-\theta,$$

$$\therefore \angle\text{CON}=\angle\text{COQ}+\angle\text{NOQ}$$

$$=90^\circ-\frac{\alpha+\beta}{2}+\beta-\theta=90+\frac{\beta-\alpha}{2}-\theta.$$

From the triangle AOC,

$$\frac{\text{CO}}{\text{AC}}=\frac{\sin\text{CAO}}{\sin\text{COA}}=\frac{\sin\theta}{\cos\left(\frac{\alpha+\beta}{2}\right)},$$

$$\therefore\ \text{CO}=r\,.\,\frac{\sin\theta}{\cos\left(\frac{\alpha+\beta}{2}\right)}.$$

Substituting these values in the foregoing equation of moments, we get

$$\text{P}\times r=(\text{P}+\text{W})r\,.\,\frac{\sin\theta}{\cos\left(\frac{\alpha+\beta}{2}\right)}\,.\,\cos\left(\frac{\beta-\alpha}{2}-\theta\right),$$

$$\therefore\ \text{P}=\frac{\text{W}\sin\theta\cos\left(\frac{\beta-\alpha}{2}-\theta\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)-\sin\theta\cos\left(\frac{\beta-\alpha}{2}-\theta\right)},$$

which is the expression required.

If $\alpha=\beta$, then

$$\text{P}=\frac{\text{W}\sin\theta\cos\theta}{\cos\alpha-\sin\theta\cos\theta}$$

$$=\frac{\text{W}\sin 2\theta}{2\cos\alpha-\sin 2\theta}.$$

In this equality P will be a minimum when the denominator is a maximum, or when $\alpha=0$; that is to say, P will be a minimum when the planes become horizontal.

CHAP. VIII.

FRICTION AND OTHER RESISTANCES OF MACHINES BORDERING ON A STATE OF MOTION.

FRICTION ON AN AXIS.

189. To determine the conditions of the state *bordering on* motion, when a body movable upon an axis, is acted upon by pressures in the same plane.

Let O be the centre of the axis turning in a circular bearing TD, and RT the direction of the resultant of the pressures; then, when the axis is about to slip or to turn round, the angle RTC must be equal to the angle of friction α.

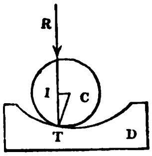


Fig. 116.

To find the work, U, in n revolutions of the axis.

Let r=CT, the radius of the axis; then,

$$\text{Circumference axis}=2r\pi;$$

Perpendicular pressure on the bearing, or in the direction CT= $\text{R}\times\cos\angle\text{CTR}=\text{R}\cos\alpha$;

$$\therefore\ \text{resistance of friction}=\text{R}\cos\alpha\times f=\text{R}\cos\alpha\times\frac{\sin\alpha}{\cos\alpha}=\text{R}\sin\alpha;$$

$$\therefore\ \text{U}=\text{R}\sin\alpha\times 2rn\pi\ \ldots (1).$$

But when α is small, which it usually is in practice, we have very nearly $\sin\alpha=\tan\alpha=f$, hence eq. (1) becomes very nearly

$$\text{U}=f\text{R}\times 2rn\pi\ \ldots (2).$$

Example. A water wheel, weighing 10,000 lbs., turns upon an an axis or gudgeon whose radius is ·25 ft.; it is required to find the work consumed by friction per minute, when the wheel makes 4 revolutions per min. and the coefficient of friction upon the axis is ·075.

Here R=the weight of the wheel=10,000, r=·25, n=4, and f=·075; therefore by eq. (2),

$$\text{U}=\cdot075\times10{,}000\times2\times\cdot25\times4\times3\cdot1416$$
$$=4712,\ \text{or about}\ \tfrac{1}{7}\ \text{of a horse power.}$$

The Lever when the Forces act perpendicular to it.

190. Let POQ be a lever having the circular axis OT, turning within a circular bearing, and maintained in equilibrium by the perpendicular pressures P and Q. The resultant of these pressures must be a vertical force R inclined to OT at the angle of friction. Taking T, therefore, as the centre of motion or fulcrum, we have by the equality of moments for the state bordering on motion in the direction of P,

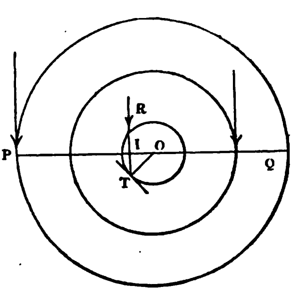


Fig. 117.

$$\text{P}\times\text{PI}=\text{Q}\times\text{QI}$$

put $PO=a$, $QO=b$, $r=OT$, and $\alpha=\angle OTR$; then

$$PI=PO-IO=a-r\sin\alpha,$$

and $QI=QO+IO=b+r\sin\alpha$;

$$\therefore\ P(a-r\sin\alpha)=Q(b+r\sin\alpha)\ldots(1);$$

$$\therefore\ P=Q\cdot\frac{b+r\sin\alpha}{a-r\sin\alpha}\ldots(2),$$

which is the equation for the state bordering on motion when P is about to preponderate.

Now, when Q is about to preponderate the resultant, R, will cut the axis to the right of O, and then we obviously have,

$$P=Q\cdot\frac{b-r\sin\alpha}{a+r\sin\alpha}\ldots(3).$$

For all values of P which lie between those given in eq. (2) and eq. (3), the lever will remain in equilibrium; these values give the limits between which motion does not take place.

In like manner, when P acts on the same side of the axis as Q, P acting upwards and Q downwards, we get,

$$P=Q\cdot\frac{b\pm r\sin\alpha}{a\pm r\sin\alpha}\ldots(4).$$

Where the + or − sign is taken according as P or Q is about to preponderate.

Example 1. In a lever POQ of the first kind, $PO=14$ in., $QO=12$ in., $OT=3$ in., $Q=5$ lbs., and $\alpha=30°$, required P when it is about to preponderate.

Here by eq. (2), we have

$$P=5\times\frac{12+3\times\frac{1}{2}}{14-3\times\frac{1}{2}}=5{\cdot}4\text{ lbs.}$$

Example 2. Required the value of P, when Q is about to preponderate.

Here by eq. (3), we have

$$P=5\times\frac{12-3\times\frac{1}{2}}{14+3\times\frac{1}{2}}=3{\cdot}4\text{ lbs.}$$

Therefore, for all values of P between 3·4 lbs. and 5·4 lbs. the lever will remain in equilibrium.

The Wheel and Axle when the Forces act vertically.

191. Let OP be the radius of the wheel, and OQ the radius of

the axle; put W for the weight of the wheel and axle, then this weight will act through the centre O; now taking T, the *actual fulcrum*, as the centre of moments, we have from *fig.* 117.,

$$P \times PI = Q \times QI + W \times OI,$$

$$\therefore\ P(a - r \sin \alpha) = Q(b + r \sin \alpha) + Wr \sin \alpha \ldots (1),$$

which is the equation for the state bordering on motion, when P is about to preponderate.

If the pressures P and Q act upwards whilst the weight W of the wheel and axle acts downwards, then it is obvious that the moment of W must have a contrary sign to that of Q; in this case, therefore, the sign of W in eq. (1) must be taken minus.

When Q is about to preponderate, we find as in the case of the lever eq. (3) Art. **190.**

$$P(a + r \sin \alpha) = Q(b - r \sin \alpha) - Wr \sin \alpha \ldots (2).$$

It will be observed, that when P and W act on opposite sides of the axis, the resultant R meets the axis on the same side as the preponderating force.

If P acts on the same side as Q, P upwards and Q downwards, then when P is about to preponderate, we have

$$P(a + r \sin \alpha) = Q(b + r \sin \alpha) + Wr \sin \alpha \ldots (3),$$

and when Q is about to preponderate, we have,

$$P(a - r \sin \alpha) = Q(b - r \sin \alpha) - Wr \sin \alpha \ldots (4).$$

Comparing eq. (3) with eq. (1), it will be readily seen that it is most advantageous to apply P after the manner described in eq. (3).

Example 1. In a wheel and axle, represented in *fig.* 117., where the power and weight both act vertically downwards, $OP = a = 24$ in., $OQ = b = 3$ in., $OT = r = 1$ in., $\alpha = 30°$; $Q = 400$ lbs. weight of the wheel and axle $= W = 40$ lbs.; required P when it is about to preponderate.

Here by eq. (1), we get

$$P(24 - 1 \times \tfrac{1}{2}) = 400(3 + 1 \times \tfrac{1}{2}) + 40 \times 1 \times \tfrac{1}{2};\ \therefore\ P = 60{\cdot}4 \text{ lbs.}$$

Example 2. Required P, in the last example, when Q is about to preponderate.

Here by eq. (2), we have

$$P(24 + 1 \times \tfrac{1}{2}) = 400(3 - 1 \times \tfrac{1}{2}) - 40 \times 1 \times \tfrac{1}{2};\ \therefore\ P = 40 \text{ lbs.}$$

Hence the limits between which equilibrium is possible are 40 and 60 lbs.

Example 3. Required the pressure P, in Example 1, when it acts on the same side of the axis as Q.

When P is about to preponderate, we have from eq. (3),

$$P(24+\tfrac{1}{2})=400(3+\tfrac{1}{2})+40\times\tfrac{1}{2}\,; \;\therefore\; P=58 \text{ lbs. nearly.}$$

This is less than the value found in example 1.

The Lever when the Forces act obliquely.

192. Given P to determine Q when the lever is about to move in the direction of the force P.

First by construction.

Let PQ represent the lever; $OT=r$ the radius of the axis; the arms of the lever, $OP=a$, and $OQ=b$; PC and QC the directions of the pressures P and Q respectively; CT the resultant of these pressures cutting the axis at T, and forming with OT the angle of friction OTC equal to α.

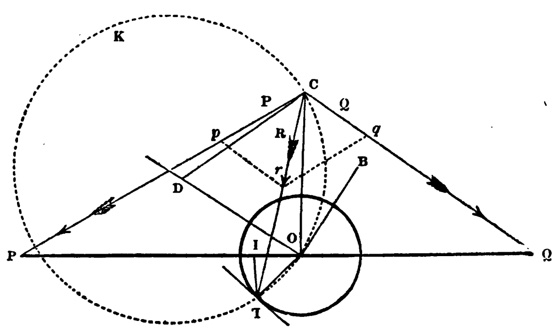


Fig. 118.

Take $PO=a$, $OQ=b$; with the radius $OT=r$, describe the circle representing the axis; draw PC and QP making the angles QPC and PQC equal to the angles at which the forces P and Q respectively act; join CO.

Now upon CO as a chord we have to describe a circle which shall contain the angle $OTC=\alpha$, or the given angle of friction. Draw OB making the angle $COB=\alpha$, the angle of friction; draw OD perpendicular to OB; from C draw CD making the angle OCD equal to the angle COD; then these two lines will intersect in a point D, which is the centre of the circle required. (See the Author's Geometry and Mensuration, p. 55., Art. **60.**, Cor. 3.)

With D as a centre and radius equal to DC or DO, describe the circle KCOT intersecting the axis of the lever in the point T; join CT; then $\angle$OTC$=\angle$COB$=\alpha$, the angle of friction; and CT will be the direction of the resultant of the two forces P and Q when the lever is in the state bordering on motion.

To determine the pressure Q: from any convenient scale, take Cp=the units of pressure in P; and construct the parallelogram of pressures Cprq; then the units in Cq will give the units of pressure of Q.

193. In the foregoing construction we have supposed that the forces act on different sides of the axis; in this case the resultant meets the axis on the same side as the preponderating force P; but when the forces act on the same side of the axis, as in *fig.* 119., the resultant TCr must cut the axis at T on the side opposite to the preponderating force P; with this difference, the construction in this latter case is precisely the same as in the former.

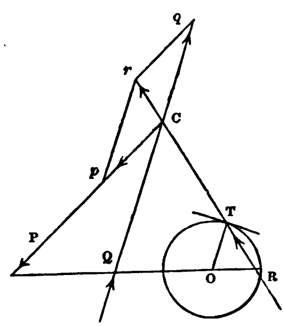


Fig. 119.

194. Second by calculation. Conceive perpendiculars to be let fall from O on the direction of the forces P, Q, and R; put p, q, and k, for these perpendiculars respectively; and ϕ for the angle PCQ in *fig.* 118., or for PCq in *fig.* 119.; then, for both cases, we have by the equality of moments, taking the direction of the force R to arise from the reaction of the axis,

$$\text{P}p-\text{Q}q=\text{R}k.$$

This holds true, for both cases, when P preponderates; but if P be upon the point of yielding, or what is the same thing, if Q be about to preponderate, then the resultant will, in both cases, lie on the other side of O; so that the equation of equilibrium now becomes,

$$\text{P}p-\text{Q}q=-\text{R}k.$$

Now if we regard k as positive or negative, according as P is about to preponderate or to yield, then we have generally,

$$\text{P}p-\text{Q}q=\text{R}k.$$

From the parallelogram of pressures Cqrp, in both figures, we have (Art. **58.**)

$$\text{R}^2=\text{P}^2+2\,\text{P}\text{Q}\cos\phi+\text{Q}^2.$$

Squaring and substituting the value of R^2, we get

$$(Pp-Qq)^2=k^2(P^2+2PQ\cos\phi+Q^2),$$

and by reduction we get

$$\left(\frac{P}{Q}\right)^2-\frac{P}{Q}\times 2\,.\,\frac{pq+k^2\cos\phi}{p^2-k^2}=-\frac{q^2-k^2}{p^2-k^2},$$

solving this quadratic for the value of $\frac{P}{Q}$, we obtain

$$\frac{P}{Q}=\frac{pq+k^2\cos\phi\pm k\sqrt{K^2-k^2\sin^2\phi}}{p^2-k^2}\quad\ldots\ (1),$$

where for the sake of conciseness K^2 is put for $p^2+2pq\cos\phi+q^2$. Here again we observe that the sign of k determines whether the $+$ or $-$ value is to be taken in the expression.

In order to obtain an approximate expression, divide the numerator and denominator by p^2, and neglect the terms involving $\left(\frac{k}{p}\right)^2$ as being very small; then

$$\frac{P}{Q}=\frac{q}{p}\pm\frac{k}{p^2}\,.\,K,$$

but $k=r\sin\alpha$,

$$\therefore\ P=\left\{\frac{q}{p}\pm\frac{r\sin\alpha}{p^2}\,.\,K\right\}Q\quad\ldots\ (2),$$

where the $+$ or $-$ sign is taken according as P is about to preponderate or to yield.

When P is about to preponderate, it will be a minimum when $K=\sqrt{p^2+2pq\cos\phi+q^2}$, is a minimum, or when $\phi=180°$; and this condition will be fulfilled when the forces P and Q act on the same side of the axis and parallel to each other.

When $p=q$, the wheel and axle becomes a single pulley, and in this case eq. (2), Art. **197.**, becomes

$$P=\left\{1+\frac{r\sin\alpha}{p^2}\,.\,K\right\}Q\quad\ldots\ (3),$$

where $K^2=2p^2(1+\cos\phi)$,

which is the relation bordering on a state of motion when P is about to preponderate.

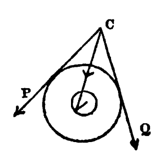


Fig. 119.*

Wheel and Axle, when its Weight is taken into Account.

195. Let the three pressures P, P_1, P_2, act as in *fig.* 120., where P_2 acting vertically, may represent the weight of the wheel and axle.

Let $\phi_2 = \angle$ PAO, the angle which P makes with the vertical; $\phi_3 = \angle P_1DO$, the angle which P_1 makes with the vertical; $\phi_1 = \phi_2 + \phi_3 = \angle PCP_1$, the angle which P and P_1 make with each other; also let a=OP, the radius of the wheel; $a_1 = OP_1$, the radius of the axle; r=O'T, the radius of the axis; $\alpha = \angle$ OTR, the angle of friction, or the angle which the resultant R makes with the radius of the axis when the machine is bordering upon motion by the preponderance of P; $\therefore$ $r \sin \alpha$ = the perpendicular on R.

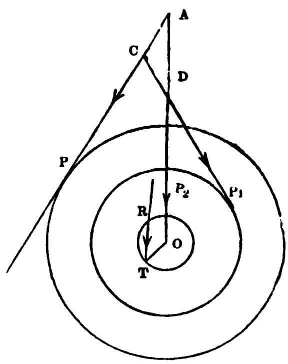


Fig. 120.

By eq. (7), Art. **65.**,

$$R^2 = P^2 + P_1^2 + P_2^2 + 2PP_1 \cos \phi_1 + 2PP_2 \cos \phi_2 + 2P_1P_2 \cos \phi_3.$$

Moreover taking O as the centre of moments, we have

$$P \cdot a - P_1 \cdot a_1 = R \cdot r \sin \alpha,$$

$$\therefore \; P^2a^2 - 2PP_1aa_1 + P_1^2a_1^2 = R^2 \cdot r^2 \sin^2 \alpha,$$

substituting the value of R^2, and reducing, we get

$$P^2 - P \cdot 2\frac{P_1aa_1 + r^2 \sin^2 \alpha\,(P_1 \cos \phi_1 + P_2 \cos \phi_2)}{a^2 - r^2 \sin^2 \alpha}$$

$$= \frac{r^2 \sin^2 \alpha\,(P_1^2 + P_2^2 + 2P_1P_2 \cos \phi_3) - P_1^2a_1^2}{a^2 - r^2 \sin^2 \alpha} \quad \ldots (1).$$

This equation may be solved as a quadratic for the value of P, which Professor Moseley has done in his "Engineering," p. 191.; but the result may be simplified by making the following assumption.

When a machine works with its greatest efficiency, the useful pressure P_1 will be a certain constant quantity which will have a fixed ratio to the weight P_2 of the wheel work; let us therefore

suppose that P_2 is the nth part of P_1, that is, let $P_2 = nP_1$. Making this substitution in eq. (1), then dividing by P_1^2, and for the sake of abbreviation putting

$$A = a a_1 + r^2 \sin^2 \alpha (\cos \phi_1 + n \cos \phi_2),$$

$$B = a^2 - r^2 \sin^2 \alpha,$$

$$\text{and } C = r^2 \sin^2 \alpha (1 + n^2 + 2n \cos \phi_3) - a_1^2\,;$$

hence we get

$$\left(\frac{P}{P_1}\right)^2 - 2 \cdot \frac{A}{B} \cdot \frac{P}{P_1} = \frac{C}{B},$$

$$\therefore \frac{P}{P_1} = \frac{1}{B}(A + \sqrt{B \cdot C + A^2}) \ldots (2),$$

which is the general relation of the pressures in the wheel axle when bordering on a state of motion.

If $n = 0$, this expression becomes the same as that given in eq. (2), Art. **194.**

Let P_1 act vertically, as in *fig.* 121., then $\phi_1 = \phi_2$, and $\phi_3 = 0$; in this case

$$A = a a_1 + r^2 \sin^2 \alpha \cos \phi_1 (1 + n)\,;$$

$$B = a^2 - r^2 \sin^2 \alpha\,;$$

$$\text{and } C = r^2 \sin^2 \alpha (1 + n)^2 - a_1^2.$$

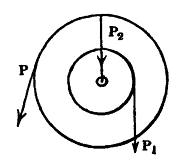


Fig. 121.

substituting in eq. (2), and omitting the terms which contain powers of r above the first, as being very small, we get

$$\frac{P}{P_1} = \frac{a_1}{a} + \frac{r \sin \alpha}{a^2} \sqrt{a^2(1+n)^2 + a_1^2 + 2 a a_1 \cos \phi_1 (1+n)} \ldots (3).$$

Or by an obvious abbreviation, this equality may be written

$$\frac{P}{P_1} = \frac{a_1}{a} + \frac{J}{a^2}.$$

If P acts on the same side as P_1 and in a direction contrary to it, as shown in *fig.* 122., then $\cos \phi_1$ becomes minus, for in this case ϕ_1 is an angle greater than 90°.

If P acts horizontally, then $\phi_1 = 90°$, and eq. (3) becomes

$$\frac{P}{P_1} = \frac{a_1}{a} + \frac{r}{a^2} \sin \alpha \sqrt{a^2(1+n)^2 + a_1^2} \ldots (4).$$

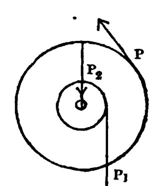


Fig. 122.

If $a_1 = a$, in eq. (4), the machine becomes a single

pulley, where the forces act as in *fig.* 123., and then we get

$$\frac{P}{P_1} = 1 + \frac{r}{a} \sin \alpha \sqrt{(1+n)^2+1} \ \ldots \ (5).$$

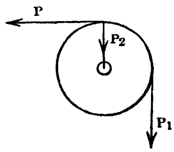


Fig. 123.

Now $\sqrt{(1+n)^2+1} = \sqrt{2}\{1+n(1+\frac{1}{2}n)\}^{\frac{1}{2}}$
$= \sqrt{2}(1+\frac{1}{2}n)$,

omitting the terms which contain the powers of n above the first. Substituting this in eq. (5), we get

$$\frac{P}{P_1} = 1 + \frac{r}{a} \sin \alpha \left(\sqrt{2} + \frac{n}{\sqrt{2}}\right),$$

substituting the value of n, that is, $n = \frac{P_2}{P_1}$, and reducing, we get

$$P = \left(1 + \frac{\sqrt{2}.r \sin \alpha}{a}\right) P_1 + \frac{r \sin \alpha}{\sqrt{2}.a} . P_2 \ \ldots \ (6).$$

Values of f or tan α, according to the Experiments of Morin.

Iron on Oak	·62
Cast Iron on Oak	·49
Oak on Oak (fibres parallel)	·48
Ditto ditto, greased	·10
Cast Iron on Cast Iron	·15
Wrought Iron on Wrought Iron	·14
Brass on Iron	·16
Brass on Brass	·20
Wrought Iron on Cast Iron	·19
Cast Iron on Elm	·19
Soft Limestone on the same	·64
Hard Limestone on the same	·38
Leather Belts on Wooden Pulleys	·47
Leather Belts on Cast Iron Pulleys	·28
Cast Iron on Cast Iron greased	·10
Brass on Iron greased	·08
Pivots or *Axes* of Wrought or Cast Iron on Brass or Cast Iron pillows,	
—— constantly supplied with oil	·05
—— greased from time to time	·08
—— without any application, or dry	·15

A more extensive table will be found in Moseley's "Engineering," page 149.

RIGIDITY OF CORDS.

196. Two weights P and Q, attached to a stiff cord going over a wheel, will balance each other when their weights are equal, but in order that P should preponderate, it should exceed Q by a pressure requisite to overcome not only the friction on the axis, but also the rigidity of the cord, or that force which is required to bend the cord over the curved surface of the wheel. From a series of experiments, Coulomb found that this force of rigidity acted so as to increase Q by the quantity $D+E \cdot Q$, where D and E are constants which he determined by experiment for ropes of various circumferences, &c. This law applies to cords bent upon wheels of equal radii; but when the radii are different the quantity $D+E \cdot Q$ varies inversely as the radii, so that if R be the radius of a wheel we have in general $\frac{D+E \cdot Q}{R}(=q)$ for the quantity by which Q is increased. Hence it appears that a weight Q acting by a stiff cord going over a wheel whose radius is R produces a force represented by

$$Q+\frac{D+E \cdot Q}{R} \ldots (1)$$

when it is in a state bordering on motion.

When Q is large, which it is always in practice, then we have very nearly $q=\frac{E \cdot Q}{R}$, and therefore the total resistance produced by the weight Q in this case will be expressed by

$$Q\left(1+\frac{E}{R}\right)\ldots (2).$$

According to Morin the formula for the resistance due to the rigidity of *new white ropes* is

$$q=\frac{n}{d}(\cdot 0002+\cdot 00017\,n+\cdot 000243\,Q),$$

and for *tarred ropes*

$$q=\frac{n}{d}(\cdot 001+\cdot 000232\,n+\cdot 00028\,Q),$$

where d=the diameter of the wheel in feet, n=the number of threads in the rope.

The values of the constants D and E are given, in a few cases, in the following Table, for different dimensions of the cords. A more extended Table is given in Moseley's "Engineering," page 158.

Table. Rigidity of Ropes.

197. Values of the constants D and E, according to the experiments of Coulomb. The radius of the wheel being taken 1 foot.

No. 1. New dry cords. Rigidity varies as the square of the circumference.

Circumference of Rope in Inches.	Value of D in lbs.	Value of E in lbs.
1	·1315	·00575
2	·5261	·02303
4	2·1044	·07317
8	8·4137	·36849

No. 2. New ropes dipped in water.

Circumference of Rope in Inches.	Value of D in lbs.	Value of E in lbs.
1	·2630	·00575
2	1·0522	·02303
4	4·2089	·07317
8	16·8356	·36849

Example 1. A dry white rope, 2 inch circumference, passes over a wheel 2 feet in diameter, from which is suspended a weight of 1000 lbs.; required the power, P, which must be applied at the other extremity of the rope in order to raise the weight, the friction on the axis of the wheel being neglected.

Here $Q=1000$, $R=\frac{1}{2}$ of $2=1$, $D=\cdot526$, $E=\cdot02303$, hence we have by the first formula

$$P=1000+\cdot526+\cdot02303\times1000=1023\cdot5 \text{ lbs.},$$

and by the second formula

$$P=1000(1+\cdot02303)=1023 \text{ lbs.}$$

Example 2. Required the same as in the last example, when the radius of the wheel is 3 in., $Q=4000$ lbs., and circum. rope$=4$ in.

By the first formula

$$P=4000+4(2\cdot1044+\cdot07317\times4000)=5179 \text{ lbs.},$$

By the second formula

$$\text{P}=4000(1+4\times\cdot07317)=5170 \text{ lbs.}$$

The wheel and axle, taking the rigidity of the ropes into account.

198. Here in eq. (1), Art. **191.**, we have to substitute $\text{Q}+\frac{\text{D}+\text{E}\cdot\text{Q}}{b}$ for Q, where b is the radius of the axle; hence we have

$$\text{P}(a-r\sin\alpha)=\left(\text{Q}+\frac{\text{D}+\text{E}\cdot\text{Q}}{b}\right)(b+r\sin\alpha)+\text{W}r\sin\alpha,$$

and by reduction, we get

$$\text{P}=\text{Q}\cdot\frac{(b+\text{E})(b+r\sin\alpha)}{b(a-r\sin\alpha)}+\frac{b\text{D}+(\text{D}\pm b\text{W})r\sin\alpha}{b(a-r\sin\alpha)}\ \ldots\ (1).$$

If D be neglected, then

$$\text{P}=\frac{\text{Q}(b+\text{E})(b+r\sin\alpha)\pm b\text{W}r\sin\alpha}{b(a-r\sin\alpha)},$$

which is the equation for the state bordering on motion; where the minus sign of W is taken when the pressures P and Q act upwards.

199. Proceeding in the same manner with eqs. (2), (3), &c., of Art. **194.**, we obtain the equation for the state bordering on motion, when the pressures do not act vertically.

For example, we have from eq. (6), Art. **195.**, by putting for P_1 its value $\text{P}_1\left(1+\frac{\text{E}}{a}\right)+\frac{\text{D}}{a}$, and reducing

$$\text{P}=\left(1+\frac{\text{E}+\sqrt{2}\cdot r\sin\alpha}{a}\right)\text{P}_1+\frac{1}{a}\left(\text{D}+\frac{r\sin\alpha}{\sqrt{2}}\cdot\text{P}_2\right)\ \ldots\ (2),$$

which is the equation for the single pulley when the pressures act as represented in *fig.* 123., and P_2 is small as compared with P_1. For the sake of abbreviation, we may write this equation as follows:—

$$\text{P}=e_1\text{P}_1+c_1\ \ldots\ (3).$$

Example 1. In the wheel and axle represented in *fig.* 117., let $\text{Q}=400$, $a=1\frac{1}{2}$ ft., $b=\frac{1}{4}$ ft., $r=\frac{1}{24}$ ft., $\sin\alpha=\cdot08$, $\therefore r\sin\alpha=\frac{1}{24}\times\cdot08=\cdot0033$, $\text{W}=100$, circumference cord=2 in. supposed to be dry and new; required P when it is about to preponderate.

Here the value of P is given in eq. (1) Art. **198.** From the Table, Art. **197.**, we find $D = \cdot526$, $E = \cdot02303$,

$$\therefore\ P = \frac{400(\frac{1}{4}+\cdot02303)(\frac{1}{4}+\cdot0033)+\frac{1}{4}\times\cdot526+(\cdot526+\frac{1}{4}\times100)\cdot0033}{\frac{1}{4}(1\frac{1}{2}-\cdot0033)}$$

$$=74\cdot5 \text{ lbs.}$$

If D and W be neglected in eq. (1), we have

$$P = \frac{400(\frac{1}{4}\ \ \cdot02303)(\frac{1}{4}+\cdot0033)}{\frac{1}{4}(1\frac{1}{2}-\cdot0033)} = 74\cdot2 \text{ lbs.}$$

When all the resistances are neglected, we have

$$P = \frac{bQ}{a} = \frac{\frac{1}{4}\times400}{1\frac{1}{2}} = 66\cdot6 \text{ lbs.}$$

Hence it appears that the useless resistances of this machine chiefly depend upon the friction on the axis, and the coefficient E of the rigidity of the cord.

Example 2. In the wheel and axle represented in *fig.* 121., and eq. (3), Art. **195.**, let $\phi_1 = 60^\circ$ the angle which the direction of P makes with the vertical, the weight $Q = 400$ lbs., $a = 1\frac{1}{2}$, a_1 or $b = \frac{1}{4}$, P_2 or $W = 100$ lbs., and so on, as in the foregoing examples.

Here by expression (1), Art. **196.**, we have

$$P_1 = Q + \frac{D+E\,.\,Q}{R} = 400 + \frac{\cdot526+\cdot02303\times400}{\frac{1}{4}} = 439;$$

$\therefore\ n = \frac{P_2}{P_1} = \frac{100}{439} = \cdot228$; and $\cos\phi_1 = \frac{1}{2}$; hence we have by substitution in eq. (3), Art. **195.**,

$$P = 439\left\{\frac{1}{6} + \frac{4\times\cdot0033}{9}\sqrt{\tfrac{9}{4}(1+\cdot228)^2+\tfrac{1}{16}+\tfrac{3}{2}\times\tfrac{1}{4}(1+\cdot228)}\right\}$$

$$=74\cdot35 \text{ lbs.}$$

which is the value of P when bordering on motion.

If D be neglected, we find $P_1 = 437$, &c., and $P = 74$ lbs., which is nearly the same as the value before determined.

If all the prejudicial resistances are neglected, then

$$P = Q\times\frac{a_1}{a} = 400\times\tfrac{1}{6} = 66\cdot6 \text{ lbs.}$$

PART III.

DYNAMICS.

CHAP. IX.

COLLISION OR IMPACT OF BODIES.

200. In treating of the collision of bodies there are two cases to be considered,—(1) that of inelastic bodies, (2) that of elastic bodies.

An *elastic* body is susceptible of compression, and regains its figure after the compressing force has ceased; but an *inelastic* body does not regain its figure. Glass, ivory, &c., are highly elastic substances, whereas soft putty and clay are almost perfectly inelastic.

Inelastic Bodies.

To find the motion of two inelastic bodies after impinging directly upon each other.

201. Let M and M_1 be the quantities of matter in the two bodies, V and V_1 their velocities before impact. Suppose the bodies to be moving in the same direction, and let M overtake M_1; then M will continue to impart motion to M_1 until they have the same velocity, and they will then move on uniformly together with this common velocity. Let v=this common velocity after impact.

By the law of action and reaction, Art. **20.**, the momentum gained by M_1 will be equal to the momentum lost by M; but we have

$$\text{Momentum gained by } M_1 = M_1(v - V_1),$$
$$\text{,, \quad lost by } M = M(V - v),$$
$$\therefore\ M_1(v - V_1) = M(V - v),$$
$$Mv + M_1 v = MV + M_1 V_1 \ \ldots\ (1),$$

that is, the momentum after impact is equal to the momentum before impact. From this equality, we get

$$v=\frac{\mathrm{M}\mathrm{V}+\mathrm{M}_1\mathrm{V}_1}{\mathrm{M}+\mathrm{M}_1} \dots (2),$$

which gives the velocity after impact.

Or putting W and W_1 in the place of the masses M and M_1 respectively, eqs. (1) and (2) become,

$$\mathrm{W}v+\mathrm{W}_1 v=\mathrm{W}\mathrm{V}+\mathrm{W}_1\mathrm{V}_1 \dots (3),$$

$$\therefore\ v=\frac{\mathrm{W}\mathrm{V}+\mathrm{W}_1\mathrm{V}_1}{\mathrm{W}+\mathrm{W}_1} \dots (4).$$

From eq. (4) we get

Momentum gained by W_1, or lost by $\mathrm{W}=\mathrm{W}_1 v-\mathrm{W}_1\mathrm{V}_1$

$$=\frac{\mathrm{W}\mathrm{W}_1(\mathrm{V}-\mathrm{V}_1)}{\mathrm{W}+\mathrm{W}_1} \dots. (5).$$

If the bodies are moving in *opposite* directions, we have merely to write $-\mathrm{V}_1$ for V_1 in these expressions.

202. *There is a loss of work by the impact of two inelastic bodies.*

$$\text{Work before impact}=\frac{\mathrm{W}\mathrm{V}^2}{2g}+\frac{\mathrm{W}_1\mathrm{V}_1{}^2}{2g};$$

$$\text{Work after impact}=\frac{(\mathrm{W}+\mathrm{W}_1)v^2}{2g}=\frac{(\mathrm{W}\mathrm{V}+\mathrm{W}_1\mathrm{V}_1)^2}{2g(\mathrm{W}+\mathrm{W}_1)},$$

by substituting the value of v given in eq. (4);

$$\therefore\ \text{Work lost}=\frac{\mathrm{W}\mathrm{V}^2}{2g}+\frac{\mathrm{W}_1\mathrm{V}_1{}^2}{2g}-\frac{(\mathrm{W}\mathrm{V}+\mathrm{W}_1\mathrm{V}_1)^2}{2g(\mathrm{W}+\mathrm{W}_1)}$$

$$=\frac{\mathrm{W}\mathrm{W}_1}{2g(\mathrm{W}+\mathrm{W}_1)}(\mathrm{V}-\mathrm{V}_1)^2 \dots (6).$$

This work lost is that which is expended in producing the compression of the bodies, for as the bodies are supposed to be inelastic, the work of compression is not reproduced by the restitution of the bodies to their original form.

If W_1 be very great as compared with W, then $\frac{\mathrm{W}_1}{\mathrm{W}+\mathrm{W}_1}=1$ very nearly, and in this case eq. (5) becomes

$$\text{Work lost}=\frac{\mathrm{W}}{2g}(\mathrm{V}-\mathrm{V}_1)^2 \dots (7).$$

203. *If the velocities of the bodies be given, and also the sum of their weights, the work lost will be a maximum when the weights of the bodies are equal.*

For the work lost will be a maximum when the product of the two weights is a maximum, the sum of the weights being constant. In this case, therefore, we have from eq. (6), by making $W_1 = W$,

$$\text{Work lost} = \frac{W}{4g}(V - V_1)^2 \;\ldots\; (8).$$

ELASTIC BODIES.

204. When two elastic bodies impinge upon each other, there are two forces called into action, viz., the force of compression and the force of restitution. At the instant of the greatest compression the bodies move together as in the case of inelastic bodies; but from the force of restitution, or that tendency which elastic bodies have to regain their original form, the one body is thrown forward with the same momentum that the other body is thrown back. When the force of restitution is equal to the force of compression the bodies are said to be perfectly elastic; and on the other hand when the force of restitution is less than that of compression, the bodies are said to be imperfectly elastic. Now it has been found that the force of restitution, in the same bodies, has a constant ratio to the force of compression, whatever may be their velocities: thus if e represent this ratio, R the momentum gained by W_1, or lost by W, from compression, then eR will be the momentum gained by W_1, or lost by W, from restitution.

To find the velocities of two elastic bodies after impact.

205. Let W, W_1 be the weights of the bodies, V and V_1 their velocities before impact, v and v_1 their velocities after impact; then

Momentum gained by W_1 or lost by W after impact

$$= R + eR = (1+e)R,$$

$\therefore$ Momentum W_1 after impact, or $W_1 v_1 = W_1 V_1 + (1+e)R$,

 „ W „ , or $Wv = WV - (1+e)R$,

$$\therefore\; v_1 = V_1 + (1+e)\frac{R}{W_1},$$

$$\text{and, } v = \text{v} - (1+e)\frac{\text{R}}{\text{W}};$$

but R is given in eq. (5), therefore by substitution and reduction, we get

$$v_1 = \text{v}_1 + (1+e)\frac{\text{W}(\text{v}-\text{v}_1)}{\text{W}+\text{W}_1} \ldots (9)$$

$$v = \text{v} - (1+e)\frac{\text{W}_1(\text{v}-\text{v}_1)}{\text{W}+\text{W}_1} \ldots (10),$$

which are the velocities required. When the bodies are moving in contrary directions, we have merely to write $-\text{v}_1$ for v_1 in these expressions.

206. When $e=1$, the bodies are perfectly elastic.

207. If W_1 is a fixed obstacle, then $\text{v}_1=0$, and $\frac{\text{W}_1}{\text{W}+\text{W}_1}=1$; in this case eq. (10) becomes

$$v = -e\text{v} \ldots (11),$$

that is, W would rebound from the obstacles with a velocity equal to e times that with which it approached.

208. *To find the relative velocities of the bodies after impact.*

Subtracting eq. (10) from (9), and reducing, we get

$$v_1 - v = e(\text{v} - \text{v}_1) \ldots (12),$$

that is to say, ***the relative velocities of the bodies after impact have the constant ratio e to their relative velocities before impact.***

Newton discovered this law by experiment, and subsequent experimentalists have found it to be very nearly correct. The foregoing investigation shows, that this experimental law may be explained by the hypothesis that the force of restitution has a constant ratio to the force of compression.

209. *To determine the work,* U, *lost by the impact of two imperfectly elastic bodies.*

Hence we have by Art. **47.**,

$$\text{Work before impact} = \frac{\text{W}\text{v}^2}{2g} + \frac{\text{W}_1\text{v}_1^2}{2g},$$

$$\text{Work after impact} = \frac{\text{W}v^2}{2g} + \frac{\text{W}_1 v_1^2}{2g},$$

$$\therefore \text{Work lost, or } u = \frac{1}{2g}(\text{W}\text{v}^2 + \text{W}_1\text{v}_1^2 - \text{W}v^2 - \text{W}_1 v_1^2);$$

substituting the values of v_1 and v given in eqs. (9) and (10), and reducing, we get

$$u = \frac{(1-e^2) \text{W} \text{W}_1 (\text{V}-\text{V}_1)^2}{2g(\text{W}+\text{W}_1)} \ldots (13).$$

When $e=0$, this expression becomes the same as that given in eq. (6).

When the elasticity of the bodies is perfect, $e=1$, and eq. (13) becomes

$$u=0,$$

that is to say, THERE IS NO WORK LOST BY THE IMPACT OF TWO PERFECTLY ELASTIC BODIES. The accumulated work yielded by the one body is taken up by the other.

IMPACT UPON FIXED PLANES.

210. If an elastic body A impinge obliquely, in the direction AB, upon a smooth plane HR, the body will be reflected in the oblique direction BC. If BE be drawn perpendicular to HR, then the angle ABE is called the *angle of incidence*, and the angle EBC the *angle of reflexion.*

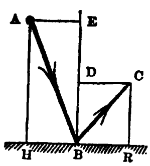


Fig. 124.

Let v=the velocity of the body before impact, v_1=the velocity after impact, e=the modulus of elasticity, $\alpha=\angle$ ABE, $\beta=\angle$ EBC; let AB represent v; draw AE parallel to HR; then from the parallelogram of motion, v is compounded of a velocity HB in the direction of the plane and BE perpendicular to it. But HB=AB $\sin\alpha=v\sin\alpha$, and BE$=v\cos\alpha$, but after impact this latter motion, from the reaction of the plane, becomes $ev\cos\alpha$. Now take BR=BH=$v\sin\alpha$, BD$=ev\cos\alpha$, and construct the parallelogram BRCD, then BC will represent the velocity of reflexion.

$$\therefore\ v_1 = \text{BC} = \sqrt{\text{BR}^2+\text{BD}^2}$$
$$= v\sqrt{\sin^2\alpha + e^2\cos^2\alpha} \ldots (1).$$

If $e=1$, that is, if the body is perfectly elastic, then $v_1=v$, and BD=BE; that is to say, *the angle of reflexion is equal to the angle of incidence.*

If the body is non-elastic, $e=0$, $\therefore$ BD=0, and $v_1=v\sin\alpha$, that is to say, *a non-elastic body, after impact, slides along the plane with the velocity* $v\sin\alpha$.

EXERCISES FOR THE STUDENT.

1. Two inelastic bodies weighing 2 lbs. and 4 lbs., and moving in the same direction, with the velocities of 6 and 9 feet respectively, impinge upon each other, required their velocity after impact. *Ans.* 8 ft.

2. Required the work lost by impact, in the last example. See Art. **202**. *Ans.* $\frac{36}{193}$.

Or thus without the formula.

$$\text{Work before impact} = \frac{2 \times 6^2}{2 \times 32\frac{1}{6}} + \frac{4 \times 9^2}{2 \times 32\frac{1}{6}} = \frac{1}{193} \times 1188$$

$$\text{Work after impact} = \frac{(2+4)8^2}{2 \times 32\frac{1}{6}} = \frac{1}{193} \times 1152,$$

$$\therefore \text{ Work lost} = \frac{1}{193}(1188 - 1152) = \frac{36}{193}$$

3. Required the same as in the last example, when the weights of the bodies are 40 and 60 lbs. and the velocities 16 and 26 feet. *Ans.* 37·3.

4. Two ivory balls, of 4 and 6 lbs. weight, impinge upon each other when moving in the same direction with the velocities of 9 and 10 ft.; required their velocities after impact, allowing the modulus of elasticity for ivory to be ·8. See eqs. (9) and (10). *Ans.* 10·08 and 9·28.

5. Show, by a particular example, that when the bodies are perfectly elastic there is no work lost by impact.

6. Two perfectly elastic balls, weighing 1 and 3 lbs., meet directly with equal velocities; show that the heavier ball will remain at rest after impact.

7. If two balls of the same substance impinge directly, show that the velocity of their centre of gravity is the same after impact that it was before impact.

In addition to the notation of Art. **204**., let V^1, $V_1{}^1$ = the velocities of the centre of gravity before and after impact; then by eq. (1), Problem 21, Art. **112**., we have

$$V^1 = \frac{W V + W_1 V_1}{W + W_1}, \text{ and } V_1{}^1 = \frac{W v + W_1 v_1}{W + W_1};$$

substituting the values of v and v_1, given in eqs. (9) and (10), in this latter equation, we get

$$v_1{}^1=\frac{w}{w+w_1}\left\{v-(1+e)\frac{w_1(v-v_1)}{w+w_1}\right\}+\frac{w_1}{w+w_1}\left\{v_1+(1+e)\frac{w(v-v_1)}{w+w_1}\right\}$$

$$=\frac{wv+w_1v_1}{w+w_1}=v^1.$$

8. Determine the motion of the centre of gravity before and after impact, when the bodies, in the last example, move in *contrary directions.*

CHAP. X.

PROJECTILES.

WHEN a body is projected obliquely to the horizon, the path which the body pursues is a curve called the parabola. By the second law of motion, gravity produces its full effect upon the body independently of its motion of projection; so that the actual path pursued by the body is compounded of the motion of projection and the motion resulting from the action of gravity.

To determine the path of a body under the action of gravity, when projected with a given velocity and in a given direction.

211. Let the body be projected in the direction AT with the given velocity v, and let t be the time which the body would take in describing the space AT, if gravity were not acting. Now if TP be the space through which the body falls in the time t; then P will be the actual place of the body.

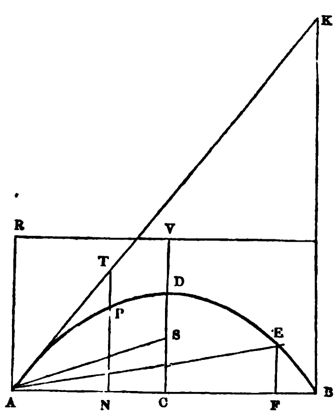


Fig. 125.

Now AT is the space which is described in the time t, with the uniform velocity v; and PT is the space which the body falls in the time t; hence we have

$$AT=vt, \text{ and } PT=\tfrac{1}{2}gt^2;$$

by eliminating t, we get

$$\text{AT}^2=\frac{2v^2}{g}.\text{PT}.$$

Let h be the height from which the body must fall in order to acquire the velocity v, then by eq. (5), Art. 26., $h=\frac{v^2}{2g}$; hence we have, by substitution,

$$\text{AT}^2=4h.\text{PT} \dots (1);$$

now this is the equation to the parabola, where AT is a tangent to the curve at A, PT is parallel to the axis CD, and h is the distance of the directrix from A, or the distance of the focus from A; thus if S be the focus then AS$=h$; if DV$=$DS, and VR is perpendicular to CV then VR is the directrix, and AR$=$AS$=h$.

To find the equation of the projectile when referred to rectangular coordinates.

212. Let $a=\angle$ BAT the angle of projection;

$$x=\text{AN}; \text{ and } y=\text{PN}; \text{ then}$$

$$\text{AT}=\text{AN} \sec a=\frac{x}{\cos a};$$

$$\text{PT}=\text{TN}-\text{PN}=x\tan a-y;$$

substituting these values in eq. (1), we get

$$\frac{x^2}{\cos^2 a}=4h\,(x\tan a-y),$$

$$\therefore\ y=x\tan a-\frac{x^2}{4h\cos^2 a} \dots (2).$$

To find the horizontal range AB.

213. Make $y=0$ in eq. (2); then the corresponding values of x will apply to the points A and B.

$$x\tan a-\frac{x^2}{4h\cos^2 a}=0,$$

whence we obtain for the roots of this equation,

$$x=0, \text{ which applies to the point A};$$

$$\text{and } x=4h\sin a\cos a$$
$$=2h\sin 2a \dots (3).$$

which is the value of AB, the horizontal range.

214. When the velocity v of projection is given, this expression for the horizontal range will be a maximum, when $\sin 2a$ is a maximum, or when $2a=90°$, or $a=45°$, that is to say, A BODY

WILL BE CARRIED TO THE GREATEST HORIZONTAL DISTANCE WHEN IT IS PROJECTED AT AN ANGLE OF 45° TO THE HORIZON.

215. Moreover, since $\sin 2(45°+\theta)=\sin 2(45-\theta)$, if we put either $45+\theta$ or $45-\theta$ for α in eq. (3) we shall have the same result; hence it appears the horizontal range is the same for any two projectiles, when the elevation of one is as much above 45° as that of the other is below it.

To find the total time of flight on a horizontal plane.

216. Draw the vertical line BK intersecting AT produced in K. Let x=the total time of flight; then

$$AK=vx, \text{ and } BK=x^2\times\frac{g}{2};$$

$$\text{but } \frac{BK}{AK}=\sin\alpha,$$

$$\therefore \frac{x^2\times\frac{g}{2}}{vx}=\sin\alpha,$$

$$\therefore x=\frac{2v\sin\alpha}{g}\ldots(4).$$

To find the greatest height.

217. It is obvious that the greatest height must be equal to the space through which the body will fall during one half of its time of flight; hence we have, by eq. (4),

$$\text{the greatest height}=\left(\frac{v\sin\alpha}{g}\right)^2\times\frac{g}{2}$$

$$=\frac{v^2\sin^2\alpha}{2g}=h\sin^2\alpha\ldots(5).$$

To find the point E where the projectile will strike an inclined plane AE passing through, A, the point of projection.

218. Let the perpendicular $EF=y$, $AF=x$, and $\angle FAE=\beta$; then

$$y=x\tan\beta;$$

but the value of y is given in eq. (2); hence we have by equality

$$x\tan\beta=x\tan\alpha-\frac{x^2}{4h\cos^2\alpha},$$

$$\therefore\ x=4h\cos^2\alpha\,(\tan\alpha-\tan\beta)$$

$$=4h\,.\,\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\beta}.$$

Now the distance AE $=x\sec\beta=\dfrac{x}{\cos\beta}$

$$=4h\,.\,\frac{\cos\alpha\sin(\alpha-\beta)}{\cos^2\beta}\ldots(6).$$

The velocity of a projectile at any point of its path is that which would be acquired in falling freely from the directrix.

219. It has been shown, Art. **211.**, that the velocity of projection is equal to the velocity which would be acquired in falling through h, the distance of the point of projection from the directrix of the parabola described. Now this must hold true for any point in the parabola, for we may obviously suppose the body to be projected, at any point of its path, with the velocity and direction which it has at that particular point.

To find the direction in which a body must be projected from a given point A, *with a given velocity, in order to hit a given mark* E.

220. Here the value of α must be found from eq. (6).

Now, $2\cos\alpha\sin(\alpha-\beta)=\sin(2\alpha-\beta)-\sin\beta$; hence we have from eq. (6), putting R for AE,

$$R=2h\,.\,\frac{\sin(2\alpha-\beta)-\sin\beta}{\cos^2\beta},$$

$$\therefore\ \sin(2\alpha-\beta)=\frac{R\cos^2\beta}{2h}+\sin\beta\ldots(7).$$

From this equation $2\alpha-\beta$ may be found, and therefore α, or the angle of projection BAT.

Exercises for the Student.

1. Prove, $\dfrac{\text{the greatest height}}{\text{horizontal range}}=\frac{1}{4}\tan\alpha.$

2. When the angle, α, of projection is equal to 15°, the horizontal range is equal to h.

3. When the angle of projection is constant, the horizontal range, as well as the greatest height, varies as the square of the velocity of projection.

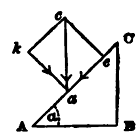
4. Show that α in eq. (7) has two values, and that they are equally inclined to a line bisecting the angle which AE makes with the vertical.

CHAP. XI.

Motion of Bodies on an Inclined Plane.

To find the relations of time, space, and velocity, when a body falls by the force of gravity down an inclined plane ABC, *the friction being neglected.*

221. Let w=the weight of the body; $\alpha = \angle$ BAC the inclination of the plane; s=AC the length of the plane; p= the pressure tending to move the body down the plane; g_1=the accelerating force which urges the body down the plane; v_1=the velocity of projection up or down the plane, as the case may be; v=the velocity thus acquired in moving through the space s.



From eq. (5), Art. **139.** we have

$$p = w \sin \alpha,$$

and from eq. (1), Art. **27.**

$$g_1 = \frac{p}{w} \cdot g$$

$$= \frac{w \sin \alpha}{w} \cdot g = g \sin \alpha \ldots (1).$$

Now, since the accelerating force g_1 is a uniform force, the equations (2), (3), and (5), given in Art. **27.** will hold true in the present case, hence we obtain, by substituting the value of g_1,

$$v = v_1 \pm g t \sin \alpha \ldots (2),$$

$$s = t v_1 \pm \tfrac{1}{2} g t^2 \sin \alpha \ldots (3),$$

$$s = \pm \frac{v^2 - v_1^2}{2 g \sin \alpha} \ldots (4),$$

and so on to any other forms of expression.

It will be observed that the + or − sign in these formulæ is to be taken according as the body is projected down or up the plane.

222. When the body simply descends the plane by the force of gravity, $v_1=0$, and then these expressions become

$$v=gt\sin\alpha \ldots (5)$$

$$s=\tfrac{1}{2}gt^2\sin\alpha \ldots (6)$$

$$s=\frac{v^2}{2g\sin\alpha} \ldots (7).$$

From eq. (7), we get

$$v^2=2gs\sin\alpha \ldots (8).$$

To find the velocity acquired in descending an inclined plane.

223. By trigonometry, we have

$$s\sin\alpha=\text{AC}\times\sin\alpha=\text{BC},$$

substituting in eq. (8), we get

$$v=\sqrt{2g\times\text{BC}};$$

but by eq. (6), Art. **26.**, this is the velocity which the body would acquire by falling freely through the vertical space BC; hence it follows, that *the velocity acquired in falling down an inclined plane, without friction, is the same as would be acquired in falling freely through the perpendicular height of the plane.*

Or thus, on the principle of accumulated work.

Work done by gravity on the body$=w\times\text{BC}$; but we have by eq. (1), Art. **47.**

Work accumulated in the body$=\dfrac{w\times v^2}{2g}$; now this accumulated work is due to the action of gravity alone,

$$\therefore \frac{w\times v^2}{2g}=w\times\text{BC},$$

$$\therefore v=\sqrt{2g\times\text{BC}},$$

which is the same result as before found.

When a body descends any arc of a smooth curve, the velocity acquired at any point is that which is due to the vertical height fallen through.

224. Let us first suppose that the body falls down a succession of smooth planes AB, BC, CD, &c., without losing any part of its acquired velocity in passing from one plane to another; then the velocity acquired at any point is due to the vertical height fallen through. Thus the velocity at B will be due to the vertical height Ab; the velocity acquired down BC will be due to the vertical height bc; so that the whole velocity at C will be due to the whole vertical height Ac; and so on.

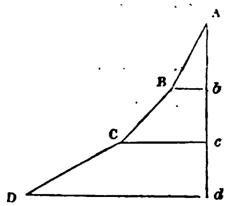


Fig. 126.

Now when the number of planes is indefinitely increased, they form a continuous curve, in which case no part of the acquired velocity is lost in passing from one part to another. Hence the velocity of the body, at any point of the curve, is that which is due to vertical height fallen through.

To find the times down any inclined planes when the height is the same.

225. From eq. (6), Art. **221.**, we have, by substituting $\frac{BC}{AC}$ for $\sin a$.

$$AC=\tfrac{1}{2}gt^2\times\frac{BC}{AC}\ldots(1)$$

$$\therefore\ t=AC\sqrt{\frac{2}{g\,.\,BC}}$$

Now when BC is constant t varies as AC, that is to say, *the times are proportional to the length.*

To show that the times of a body falling down the chords of a circle, AC *and* KC, *drawn from the extremities of the vertical diameter* AK, *are the same.*

226. Let the body descend the plane AC; from C draw CD perpendicular to AK; then AD will be the perpendicular height of the plane, and hence we find from eq. (1), Art. **225.**,

$$t^2=\frac{2}{g}\cdot\frac{AC^2}{AD};$$

but from the property of the circle, $\frac{AC^2}{AD}=AK$,

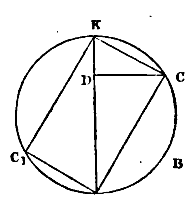


Fig. 127.

$$\therefore\ t=\sqrt{\frac{2\,\mathrm{AK}}{g}};$$

but this also expresses the time in falling freely down the diameter AK, therefore the time down any chord is the same as that down the diameter, and the time down any chords drawn from A must be the same. The same may be shown to hold for all chords drawn from K, as for example KC, KC_1, &c.

When bodies fall down any arcs, ABC, *of a circle* (see fig. 127.) *the velocities acquired at the lowest point* A *are proportional to the chords,* AC, *of the arcs.*

227. Let v=the velocity acquired in falling down the arc CBA, then, by Art. **223.**,

$$v^2=2g\times\mathrm{AD}$$

$$=2g\times\frac{\mathrm{AC}^2}{\mathrm{AK}};$$

$$\therefore\ v=\mathrm{AC}\sqrt{\frac{2g}{\mathrm{AK}}}$$

Now since AK is constant, it follows that the velocity is proportional to the length of the chord.

If the arcs are small, the velocities are very nearly proportional to the lengths of the arcs.

Exercises.

1. What velocity would a body acquire in falling down a smooth plane whose perpendicular height is 3 feet? (See Art. **223.**)

Ans. $\sqrt{193}=13{\cdot}8$ ft.

2. In what time would a body fall down a smooth plane whose length is 8 ft. and perpendicular height 4 ft.?

Here in eq. (6), Art. **221.**, we have $s=8$, $g=32\frac{1}{6}$, $\sin\alpha=\frac{4}{8}=\frac{1}{2}$, and

$$\therefore\ t=\sqrt{\frac{2s}{g\sin\alpha}}=\sqrt{\frac{16}{32\frac{1}{6}\times\frac{1}{2}}}=1 \text{ sec. nearly.}$$

3. What space would the body in the last example move over in order to acquire a velocity of 10 ft. per second? *Ans.* $3{\cdot}1+$ft.

4. What velocity will a body gain per second in falling down an inclined plane whose inclination is 45°? *Ans.* $g\sqrt{\frac{1}{2}}$.

5. What must be the inclination of an inclined plane, so that a

body falling down it may acquire a velocity of $\frac{g}{2}$ ft. in every second? *Ans.* 30°.

6. Divide the length of a given inclined plane into two parts, so that the times of descent down each of them may be the same.

Ans. The parts will be in the ratio of 1 to 3.

7. P falling down the inclined plane AC draws W up the inclined plane DC by means of a cord going over a pulley C at the common vertex of the two planes; it is required to find the velocity, v, acquired by P after descending h feet on the plane.

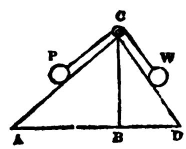


Fig. 128.

Let α=the inclination of the plane AC, and β=the inclination of DC; then

$$\text{Vertical space moved over by P} = h \sin \alpha,$$

$$\text{,, ,, ,, ,, W} = h \sin \beta;$$

$$\therefore \text{ Work due to gravity} = \text{P}h \sin \alpha - \text{W}h \sin \beta.$$

$$\text{Work accumulated in P and W} = \frac{v^2(\text{P}+\text{W})}{2g}.$$

$$\therefore \frac{v^2(\text{P}+\text{W})}{2g} = \text{P}h \sin \alpha - \text{W}h \sin \beta;$$

$$\therefore v = \sqrt{\frac{2gh}{\text{P}+\text{W}}(\text{P} \sin \alpha - \text{W} \sin \beta)}.$$

If $\alpha=\beta=90°$, this expression becomes the same as that given in Prob. (6), Art. 49.

Having found the velocity we may readily find the time.

For the space h will be described with the mean velocity $\frac{v}{2}$, hence we have

$$t = \frac{h}{\frac{1}{2}v} = \frac{2h}{v}.$$

8. An inclined plane has a rise of 5 in 100, through what space S must a body P fall down it to acquire the velocity g? Required also the time.

$$\text{Work accumulated in P} = \frac{g^2 \times \text{P}}{2g} = \frac{g}{2} \cdot \text{P};$$

$$\text{Vertical space fallen} = \frac{5}{100} \times \text{S};$$

$$\therefore \text{ Work due to gravity} = \frac{5}{100} \times s \times P;$$

$$\therefore \frac{5}{100} \times s \times P = \frac{g}{2} \cdot P;$$

$$\therefore s = 10g.$$

Moreover, $t = \dfrac{s}{\frac{1}{2}g} = \dfrac{2 \times 10g}{g} = 20$ seconds.

9. If a body be projected obliquely on an inclined plane, the path of the body is a parabola.

General Propositions* relative to the Motion of a Body on an Inclined Plane, the Friction being given.

228. A body is moved up or down an inclined plane ABC (see *fig.* 77.) by a pressure P acting parallel to the plane; required the accumulated work and the relations of the various elements, as in problems 11 and 12., Art. **49.**

Now we have in this general problem, as well as in all cases of machinery,

Work due to P = work due to friction $\pm$ work due to gravity + work accumulated in the mass.

Let $s = AC$, the space moved over by P; v_1 = the velocity of the body at first, and v the velocity after it has moved over s ft.; U_1 = the work due to friction; U_2 = the work due to gravity; u = the work accumulated or the work gained or lost; then

$$P \cdot s = U_1 \pm U_2 + u \ldots (1);$$

where the body is supposed to be moving in the direction of P, and the + or − sign is taken according as the pressure P acts up or down the plane.

By eqs. (1) and (2), Art. **170.**,

$$U_1 + U_2 = f \cdot w \cdot AB \pm w \cdot BC$$
$$= sw(f \cos \alpha \pm \sin \alpha).$$

Substituting in eq. (1), we get

$$u, \text{ or } \frac{w(v^2 - v_1^2)}{2g} = s\{P - w(f \cos \alpha \pm \sin \alpha)\} \ldots (2),$$

* First given by the Author in the "Mechanics' Magazine."

from which equation any one of the elements may be found when all the others are given. Let us derive this result by another process.

229. By eq. (1), Art. **173.**,

The moving pressure, $p=\text{P}-w(f\cos\alpha\pm\sin\alpha)\ldots(3)$.

By substituting this value of p in eqs. (6), (7), and (9), Art. **27.**; we obtain from (6) the relation between the velocity and time, from (7) the relation between the space and time, and from (9) the equality (2) just found.

When the Body is acted upon by the Force of Gravity alone.

230. If $\text{P}=0$, and the body be projected down the plane; then eqs. (3) and (2) become

$$p=w(\sin\alpha-f\cos\alpha)\ldots(4),$$

$$\frac{v^2-v_1^2}{2g}=\text{S}(\sin\alpha-f\cos\alpha)$$

$$=\text{BC}-f.\text{AB}\ldots(5).$$

And when the body is projected up the plane,

$$p=-w(\sin\alpha+f\cos\alpha)\ldots(6)$$

$$\frac{v_1^2-v^2}{2g}=\text{S}(\sin\alpha+f\cos\alpha)$$

$$=\text{BC}+f.\text{AB}\ldots(7).$$

The value of p in eq. (6) is always a retarding pressure, whereas in eq. (4) it will be an accelerating pressure when $\sin\alpha$ is greater than $f\cos\alpha$.

231. Eqs. (4) and (6) may be written in one expression, thus

$$p=-w(f\cos\alpha\pm\sin\alpha)\ldots(8);$$

and, in like manner, putting θ for the angle of friction, eqs. (5) and (7) may be written

$$\frac{v^2-v_1^2}{2g}=-\text{S}(f\cos\alpha\pm\sin\alpha)\ldots(9)$$

$$=-(\pm\text{BC}+\tan\theta.\text{AB}),$$

where the + or − sign is taken according as the body moves up or down the plane.

232. Eqs. (8) and (9) may be put in a more concise form; for we have

$$f \cos \alpha \pm \sin \alpha = \frac{\sin \theta \cos \alpha \pm \cos \theta \sin \alpha}{\cos \theta}$$

$$= \frac{\sin (\theta \pm \alpha)}{\cos \theta};$$

substituting this in eqs. (8) and (9), we get

$$p = -\frac{w \sin (\theta \pm \alpha)}{\cos \theta} \ldots (10),$$

$$\text{and } \frac{v^2 - v_1^2}{2g} = -\frac{s \sin (\theta \pm \alpha)}{\cos \theta} \ldots (11).$$

To find the relation of Space and Velocity.

233. From eq. (11) we get

$$s = -\frac{v^2 - v_1^2}{2g} \cdot \frac{\cos \theta}{\sin (\theta \pm \alpha)} \ldots (1).$$

When the body falls down the inclined plane by its own weight, $v_1 = 0$, and the minus sign of α is taken, in this case eq. (1) becomes

$$s = \frac{v^2}{2g} \cdot \frac{\cos \theta}{\sin (\alpha - \theta)} \ldots (2).$$

From this equality, we get

$$v^2 = 2gs \cdot \frac{\sin (\alpha - \theta)}{\cos \theta} \ldots (3).$$

To find the accelerating Force g_1.

234. From eq. (1), Art. **27.**, and eq. (10), Art. **232.**, we get

$$g_1 = \frac{p}{w} \cdot g = -\frac{g \sin (\theta \pm \alpha)}{\cos \theta} \ldots (1).$$

Or from eq. (1), Art. **27.**, and eq. (8), Art. **231.**, we get

$$g_1 = \frac{p}{w} \cdot g = -g(f \cos \alpha \pm \sin \alpha) \ldots (2).$$

To find the relations of space and time, when a body falls down the inclined plane AC. (See *fig.* 129.)

235. Here, we have $AC = t^2 \times \frac{g_1}{2}$.

Substituting the values of g_1, given in eqs. (1) and (2), Art. 234., observing that in the present case the minus signs are taken, we get

$$\text{AC}=t^2\times\frac{g}{2}(\sin\alpha-f\cos\alpha)\ \ldots\ (1),$$

$$\text{or AC}=t^2\times\frac{g}{2}\cdot\frac{\sin(\alpha-\theta)}{\cos\theta}\ \ldots\ (2).$$

From eq. (2), we get

$$t=\sqrt{\frac{2\,\text{AC}}{g}\cdot\frac{\cos\theta}{\sin(\alpha-\theta)}}\ \ldots\ (3).$$

If $\theta=0$, then

$$t=\sqrt{\frac{2\,\text{AC}}{g\sin\alpha}}\ \ldots\ (4),$$

which is the same result as that determined in Art. 225.

If the body be projected up the plane with the velocity, v, to find the time, t, at which it will come to a state of rest.

236. In this case, we have from eq. (1), Art. 234,

$$g_1=-\frac{g\sin(\theta+\alpha)}{\cos\theta},$$

$$\therefore\ t=\frac{v}{-g_1}=\frac{v}{g}\cdot\frac{\cos\theta}{\sin(\theta+\alpha)}\ \ldots\ (1)$$

where $\frac{v}{g}$ is the time in which the motion would be destroyed if gravity were acting freely on the body.

This expression also gives the time in which the body would acquire the given velocity v by descending the plane.

237. If the body be projected on the horizontal plane, then $\alpha=0$, and eq. (1) becomes

$$t=\frac{v}{g}\cdot\frac{1}{\tan\theta}=\frac{v}{g}\cdot\frac{1}{f}\ \ldots\ (1).$$

In order to give a geometrical form to this result; take the units in the vertical AK (see *fig.* 130.) equal to the units of seconds required for a falling body to acquire the velocity v; draw KC cutting the horizontal plane in C and making the angle AKC equal to the complement of the angle of friction; then the units in AC will give the units of seconds before the body stops.

To find the velocity gained or lost by a body moving on an inclined plane, ABC, *whose coefficient of friction is given.*

238. Here the velocity gained or lost may be found from eq. (9), Art. **231.**

In order to give a geometrical interpretation to this result; draw CH parallel to AB, and AH to BC; also draw CK and CK_1, making the angles HCK and HCK_1, respectively, equal to θ, the angle of friction; then HK or $HK_1 =$ $\tan\theta$. AB.

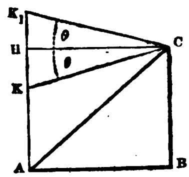


Fig. 129.

Let the body be projected down the plane; then

$$\frac{v^2-v_1^2}{2g}=\text{BC}-\tan\theta\,.\,\text{AB}$$

$$=\text{BC}-\text{HK}$$

$$=\text{AK}\;\ldots\;(1),$$

that is to say, *the velocity gained by the body in descending from* C *to* A *is equal to the velocity which it would acquire in falling freely through* AK. See eq. (7), Art. **26.**

Let the body be projected up the plane; then

$$\frac{v_1^2-v^2}{2g}=\text{BC}+\tan\theta\,.\,\text{AB}$$

$$=\text{BC}+\text{HK}_1$$

$$=\text{AK}_1\;\ldots\;(2),$$

that is to say, *the velocity lost by the body in ascending from* A *to* C *is equal to the velocity which it would lose in ascending freely through* AK_1.

Cor. 1. When the angle of friction is equal to 0, then the velocity gained or lost is simply due to AH, the vertical height of the plane, which is a well known dynamical theorem. See Art. **223.**

Cor. 2. If the inclination of the plane be equal to the angle of friction; then HK$=$BC, and AK$=0$, and therefore, in this case, the body will move *uniformly* down the plane with the velocity of projection.

To find the point to which a body will ascend an inclined plane, ABC, *when the velocity of projection is given.*

239. Take the vertical AK_1 to represent the height from which a body must fall in order to acquire the given velocity of projec-

tion; draw K_1C, cutting the plane in C, and making the angle AK_1C equal to the complement of the angle of friction; then C will be the point to which the body will ascend.

For in this case, from eq. (2), we have

$$v=0, \text{ and } \frac{v_1^2}{2g}=AK_1.$$

COR. 1.—*When the plane* AC *is horizontal.*

Take the vertical AK to represent the height from which the body must fall to acquire the velocity which it has at A; draw KC, cutting the horizontal plane in C, and making the angle AKC equal to the complement of the angle of friction; then C will be the point at which the body will come to a state of rest.

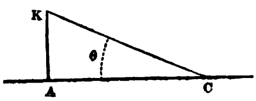


Fig. 130.

To find the distance, AQ, *which a body will move over, on the horizonal plane* AD *after descending the inclined plane* AC.

240. Draw CK, as in Art. 238., making the angle HCK equal to the angle of friction; produce CK until it intersects AD in Q; then AQ will be the space which the body will move over before it comes to a state of rest.

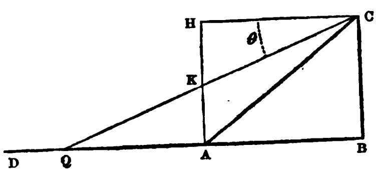


Fig. 131.

For AK will be the vertical height due to the velocity acquired in falling down AC, and since $\angle AQC=\angle HCK$, therefore by Cor. 1, Art. 239., Q will be the point at which the body will come to a state of rest.

To find the locus of the planes of equal velocities.

241. Let the vertical AK represent the height from which a body must fall to acquire the given velocity; draw KCC_1 making, with the horizontal line CH, the angle KCH equal to the angle of friction; then KCC_1, &c., will be the locus of the extremities of the planes AC, AC_1, &c., of equal velocities, that is to say, the velocities acquired by a body descending these planes will be equal to the velocity acquired by a body in falling freely through AK.

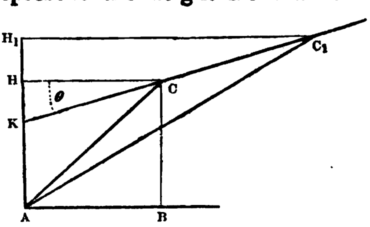


Fig. 132.

From C_1 draw C_1H_1 parallel to CH; then by Art. 238. the velocity acquired in descending the plane AC_1, or the plane AC, is equal to the velocity which would be acquired in falling freely through AK; and the same may be shown to hold true for any other plane drawn from A to meet the straight line KCC_1.

To find the locus of the planes of equal times of descent, the coefficient of friction being given.

242. Let AC, AC_1, &c., be the planes down which a body will descend in equal times. Take the vertical AK $(=2a)$ to represent the space through which the body will fall in the given time t; and let $AB=x$, $BC=y$, being the co-ordinates of C: then we have from eq. (1), Art. 235.,

$$AC^2=t^2\times\frac{g}{2}(AC\sin\alpha-f.AC\cos\alpha)$$

$$=t^2\times\frac{g}{2}(BC-f.AB).$$

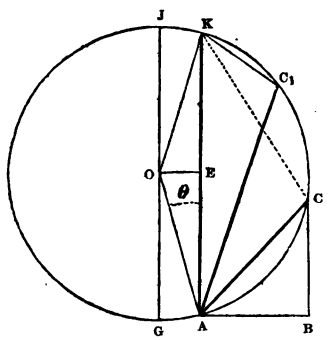


Fig. 133.

Now AK is the space through which the body will fall in the time t, therefore $AK=t^2\times\frac{g}{2}$; and from the right-angled triangle ABC, we also have $AC^2=AB^2+BC^2$; hence we find by substitution

$$AB^2+BC^2=AK(BC-f.AB),$$

$$\therefore\ x^2+y^2=2ay-2fax;$$

substituting $y+a$ for y, and $x-fa$ for x, in order to eliminate the first powers of x and y, we get

$$x^2+y^2=a^2+f^2a^2,$$

which is the equation of a circle whose radius is equal to $a\sqrt{1+f^2}$.

Let O be the centre of the circle; draw the vertical GOJ and let fall OE perpendicular to AK; then

$$OA^2=a^2+f^2a^2$$

$$=AE^2+AE^2\tan^2\theta;$$

$$\therefore\ OE=AE\tan\theta;$$

$$\therefore\ \tan\theta=\frac{OE}{AE};$$

$$\therefore \angle KAO=\theta.$$

Hence it follows that *the locus of the planes of equal times of descent is an arc of a circle whose chord is* AK, *making with the radius* OA *an angle* KAO *equal to the angle of friction.*

In like manner it may be shown that KC, KC_1, &c., are planes of equal times of descent.

COR. 1. If the planes be perfectly smooth, then $\theta=0$, and AK becomes the diameter of the circle, which is a well known dynamical theorem. (See Art. 226.)

To find the plane, AC, *of quickest descent, which can be drawn from a given point* A *to meet a given plane* BD.

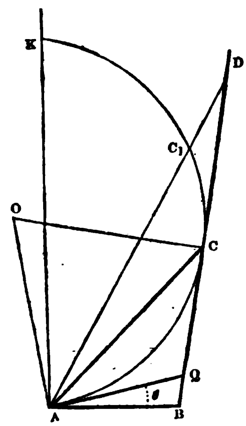


Fig. 134.

243. Let AB be a horizontal line. From A draw AQ, making the angle BAQ equal to the angle of friction; from Q take QC equal to QA; and join AC; then AC will be the plane of quickest descent.

Draw CO perpendicular to BD, and AO to AQ; then O will be centre of a circle, ACK, touching the lines BD and AQ in the points C and A. Draw the vertical AK, and join AD, cutting the circle in C_1; then $\angle OAQ=\angle KAB$, and $\therefore \angle KAO=\angle BAQ=$the angle of friction; hence it follows, Art. 242., that the times of descent down the chords AC, AC_1, &c., will be equal; therefore the time of descent down AC must be less than it will be down AD, or any other line that can be drawn from A to meet the given plane BD.

COR. 1. If the plane be perfectly smooth, then $QB=0$, and therefore $\angle BAC=\angle BCA$.

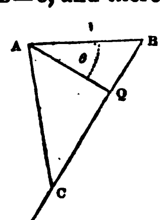


Fig. 135.

COR. 2. If the plane BD be vertical, and the angle of friction equal to nothing, then the plane AC will make with the horizon an angle of 45°. In order, therefore, to secure the most rapid descent of water, &c., from roofs, &c., the pitch should exceed 45°.

COR. 3. In the foregoing investigation, the given point, A, is assumed to be at the foot of the plane; but if the given point be assumed to be at the top of the plane, then the line AQ is drawn as in *fig.* 135.

EXERCISES AND PRACTICAL APPLICATIONS.

1. Required the solution of Problem 13., Art. 49., by the foregoing formulæ.

Making $v_1=0$. in eq. (9), Art. 231., and observing that the body moves down the plane, we get

$$\frac{v^2}{2g}=\text{BC}-f\,.\,\text{AB}.$$

Now as the inclination of the plane is small, we take the length of the plane in the place of its base, that is, we may put AC in the place of AB (see *fig.* 9.). In this formula, we have

$$\text{BC}=h,\ \text{AC or AB}=\text{S, and } f=\frac{p}{2240},$$

$$\therefore\ \frac{v^2}{2g}=h-\frac{p}{2240}\,.\,\text{S},$$

$$\therefore\ v=\sqrt{2gh-\frac{1}{1120}\cdot gp\text{S}}\ \ldots\ (1).$$

If the time, t, in descending the plane be required, we have

$$t=\frac{\text{S}}{\frac{1}{2}v}=\frac{2\text{S}}{v},$$

substituting the value of v, we get

$$t=\frac{2\text{S}}{\sqrt{2gh-\frac{1}{1120}gp\text{S}}}\ \ldots\ (2).$$

2. To solve Problem 16., Art. 49., by the foregoing formulæ.

In this case, we have from eq. (9), Art. 231.,

$$\frac{v_1^2}{2g}=\text{BC}+f\,.\,\text{AB}.$$

Here $\text{AB}=x$, $\text{BC}=\frac{ex}{100}$, $f=\frac{p}{2240}$, hence this equation becomes

$$\frac{v_1^2}{2g}=\frac{ex}{100}+\frac{px}{2240},$$

$$\therefore\ x=\frac{1120v_1^2}{g(22{\cdot}4e+p)}.$$

In this case we have for the time before the train comes to a state of rest

$$t=\frac{x}{\frac{1}{2}v_1}=\frac{2x}{v_1}$$

$$=\frac{2240\,v_1}{g(22{\cdot}4\,e+p)},$$

by substituting the value of x.

CHAP. XII.

MOTION OF ROTATION. WORK IN A ROTATING BODY. CENTRE OF GYRATION. MOMENT OF INERTIA, &c.

244. PROB. Two bodies w_1 and w_2 (whose volumes are supposed to be very small) are connected by a rod, which is made to revolve upon a centre C, the distance of w_1 from the axis is r_1 ft., and that of w_2 is r_2 ft.; if a point in the rod, at 1 foot from the axis, has a velocity of v ft. per second; it is required to determine the work, u, accumulated in the bodies, and also the point, K, in the rod where we may suppose the weight of the two bodies collected, so that the work may not be altered.

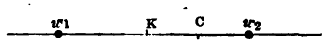


Fig. 136.

$$\text{Velocity } w_1=v\,r_1, \text{ and velocity } w_2=v\,r_2,$$

$$\therefore \text{ Work acc. in } w_1=\frac{(v\,r_1)^2\times w_1}{2g},$$

$$\text{and work acc. in } w_2=\frac{(v\,r_2)^2\times w_2}{2g},$$

$$\therefore\ u=\frac{(v\,r_1)^2\times w_1}{2g}+\frac{(v\,r_2)^2\times w_2}{2g}$$

$$=\frac{v^2}{2g}(w_1r_1^2+w_2r_2^2)\ \ldots\ (1).$$

Let m_1 be put for the volume of w_1, and m_2 for that of w_2; w=the weight of each unit of volume; then $w_1=m_1w$, and $w_2=m_2w$. By substituting these values in eq. (1), we get

$$u=\frac{v^2w}{2g}(m_1r_1^2+m_2r_2^2)\ \ldots\ (2).$$

Here the expression within the brackets is THE SUM OF THE VOLUME OF EACH BODY MULTIPLIED INTO THE SQUARE OF THE DISTANCE FROM THE AXIS; this expression is called THE MOMENT OF INERTIA of the bodies, their densities being the same, and it is usually represented by the symbol I.

Put $k=$ CK, the distance of the point K from the axis of motion; $m=$the sum of the volumes of the bodies; then

$$\text{Velocity K}=vk, \text{ and } w_1+w_2=mw;$$

$\therefore$ Work acc. in the bodies collected in the point K, or

$$u=\frac{(vk)^2\times(w_1+w_2)}{2g}=\frac{v^2w}{2g}\cdot mk^2\ldots(3),$$

but this is equal to the work expressed in eq. (2),

$$\therefore\ \frac{v^2w}{2g}\cdot mk^2=\frac{v^2w}{2g}(m_1r_1^2+m_2r_2^2),$$

$$\therefore\ mk^2=m_1r_1^2+m_2r_1^2,$$

$$\therefore\ k=\sqrt{\frac{m_1r_1^2+m_2r_2^2}{m}}\ldots(4).$$

Now the point K is called THE CENTRE OF GYRATION of the bodies, and the value of AK or k just determined, is the distance of the centre of gyration from the axis of rotation.

245. Generally let w_1, w_2, w_3, &c., be the weights of any number of bodies, at the distances r_1, r_2, r_3, &c., from the axis of rotation; k the distance of the centre of gyration of the bodies from the axis of rotation; I the moment of inertia; and so on, to the other symbols; then proceeding exactly as in the foregoing investigation, we get

$$u=\frac{v^2}{2g}(w_1r_1^2+w_2r_2^2+\&c.)\ldots(5),$$

$$u=\frac{v^2w}{2g}(m_1r_1^2+m_2r_2^2+\&c.)$$

$$=\frac{v^2w}{2g}\cdot \text{I}\ldots(6),$$

where I is put for the moment of inertia, $m_1r_1^2+m_2r_2^2+\&c.$

$$u=\frac{v^2}{2g}\cdot k^2(w_1+w_2+\&c.)\ldots(7)$$

$$=\frac{v^2w}{2g}\cdot mk^2$$

$$mk^2=m_1r_1^2+m_2r_2^2+\&c.=\text{I}\ldots(8)$$

246. When the centre of gyration in a rotating body is known, we can then readily find the accumulated work by eq. (7). But to find this point generally requires the aid of the integral calculus. The distance of the centre of gyration from the axis, in a few of the most useful cases is as follows: in a circular wheel of uniform thickness, it is equal to the radius of the wheel $\times \sqrt{\frac{1}{2}}$; in a rod revolving about its extremity, it is equal to the length of the rod $\times \sqrt{\frac{1}{3}}$, and when it revolves about its centre, it is equal to the length $\times \sqrt{\frac{1}{12}}$; and in a plane ring, like the rim of a fly wheel, it is equal to the square root of one half of the sum of the squares of the radii forming the ring. See the Author's Treatise on "The Principles of the Differential and Integral Calculus," p. 237.

247. PROB. *Given the moment of inertia,* I, *of any system of bodies* m_1, m_2, &c., *about an axis* CG *passing through the centre of gravity, to find the moment of inertia,* I_1, *about any axis* AB *parallel to the axis* CG.

Let the distances of the bodies, m_1, m_2, &c., from the axis CG be r_1, r_2, &c., and from the axis AB be R_1, R_2, &c.; then

$$I = m_1 r_1^2 + m_2 r_2^2 + \&c.$$

$$I_1 = m_1 R_1^2 + m_2 R_2^2 + \&c. \ldots, (9).$$

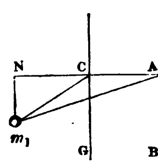


Fig. 137.

Conceive a plane to be drawn through the parallel axes CG and AB; and also another plane, m_1ACN, to be drawn, through the body m_1, perpendicular to these axes. Let m_1N be perpendicular to AC produced, and put $Cm_1 = r_1$, $Am_1 = R_1$, $AC = a$, and $CN = p_1$; then we have, by geometry,

$$Am_1^2 = Cm_1^2 + AC^2 + 2AC \cdot CN;$$

that is,
$$R_1^2 = r_1^2 + a^2 + 2ap_1.$$

In like manner we get $R_2^2 = r_2^2 + a^2 + 2ap_2$, $R_3^2 = r_3^2 + a^2 + 2ap_3$, and so on. Hence we get by substituting in eq. (9),

$$I_1 = m_1(r_1^2 + a^2 + 2ap_1) + m_2(r_2^2 + a^2 + 2ap_2) + \&c.$$
$$= m_1 r_1^2 + m_2 r_2^2 + \&c. + a^2(m_1 + m_2 + \&c.)$$
$$+ 2a(m_1 p_1 + m_2 p_2 + \&c.).$$

But because CG passes through the centre of gravity of the system, we have, Art. **89.**, $m_1 p_1 + m_2 p_2 + \&c. = 0$; putting, therefore, $m_1 + m_2 + \&c. = m$ the whole volume of the bodies, we get

$$I_1 = I + a^2 m \ldots (10);$$

that is to say, THE MOMENT OF INERTIA ABOUT ANY AXIS AB, IS EQUAL TO THE MOMENT OF INERTIA ABOUT A PARALLEL AXIS, CG,

PASSING THROUGH THE CENTRE OF GRAVITY, TOGETHER WITH THE MOMENT OF INERTIA OF ALL THE BODIES, COLLECTED IN THEIR COMMON CENTRE OF GRAVITY, ABOUT AB.

248. If k be put for the radius of gyration corresponding to the axis CG, and k_1 for that of the axis AB; then we have $I = mk^2$, and $I_1 = mk_1^2$; and by substituting these values in eq. (10), and dividing by m, we get

$$k_1^2 = k^2 + a^2 \ldots (11).$$

From the preceding equality we get

$$k_1^2 - a^2 = k^2;$$

but for the same body k must be constant,

$$\therefore\ k_1^2 - a^2 = \text{a constant};$$

that is to say, THE SQUARE OF THE RADIUS OF GYRATION, ESTIMATED FOR ANY AXIS, MINUS THE SQUARE OF THE DISTANCE OF THAT AXIS FROM THE CENTRE OF GRAVITY, IS ALWAYS A CONSTANT QUANTITY.

Ex. 1. When a uniform bar, l feet long, revolves round its middle point, the distance of the centre of gyration from this centre is $\frac{l}{\sqrt{12}}$. Required the radius of gyration, k_1, when the rod revolves round an axis at the distance of a feet from the centre.

Here by eq. (11), we have

$$k_1^2 = k^2 + a^2$$
$$= \frac{l^2}{12} + a^2,$$
$$\therefore\ k_1 = \frac{1}{2\sqrt{3}}(l^2 + 12a^2)^{\frac{1}{2}}.$$

If the rod revolves round its extremity, then $a = \frac{1}{2}l$, and $\therefore\ k_1 = \frac{l}{\sqrt{3}}$.

If $a = l$, then $k_1 = l\sqrt{\frac{13}{12}}$.

Ex. 2. Required the radius of gyration of an eccentric wheel, whose radius is r, and distance of the axis from the centre a.

In this case, $k = r \times \sqrt{\frac{1}{2}}$, see Art. **246.**;

$$\therefore\ k_1^2 = k^2 + a^2$$
$$= \frac{r^2}{2} + a^2,$$
$$\therefore\ k_1 = \sqrt{\tfrac{1}{2}(r^2 + 2a^2)}.$$

If the axis is placed at the edge of the wheel, then $a=r$, and $\therefore\ k_1=r\times\sqrt{\frac{3}{2}}$.

249. *Given the radii of gyration of the parts of a body to find the radius of gyration of the whole body.*

In eq. (8), Art. **245.**, substituting k_1 for r_1, k_2 for r_2, and so on, we get

$$mk^2=m_1k_1^2+m_2k_2^2+\&c. \dots (12),$$

where k is the radius of gyration of the whole mass m, k_1 that of m_1, and so on. If the body be of uniform thickness and the plane of rotation corresponds with the surface of the body, as in the case of a flat wheel, then the surfaces of the parts may be taken as the volumes.

To find the Radius of Gyration of a Fly Wheel.

250. Given the radius of gyration of a circular wheel of uniform thickness, revolving on its centre, to find that of a circular ring.

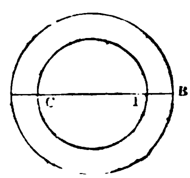


Fig. 138.

Let R = the radius of the outer circle AB; r = the radius of the inner circle CD; k and k_1 being put for the radii of gyration of these circles respectively; and k_2 for that of the circular ring; then we have, by eq. (12),

$$mk^2=m_1k_1^2+m_2k_2^2,$$

where we have, in this case, m = area cir. AB $=\pi R^2$; m_1 = area cir. CD $=\pi r^2$; and m_2 = area cir. ring $=\pi(R^2-r^2)$; moreover we have given, $k=R\sqrt{\frac{1}{2}}$, and $k_1=r\sqrt{\frac{1}{2}}$; hence we get, by substitution and an easy reduction,

$$R^4\times\tfrac{1}{2}=r^4\times\tfrac{1}{2}+(R^2-r^2)k_2^2,$$

$$\therefore\ k_2=\sqrt{\tfrac{1}{2}(R^2+r^2)} \dots (13).$$

If R_1 = the mean radius, and e = the depth of the rim, then $R=R_1+\frac{e}{2}$, and $r=R_1-\frac{e}{2}$; substituting in eq. (13), we get

$$k_2=\sqrt{R_1^2+\tfrac{1}{4}e^2} \dots (14).$$

For most practical purposes, this gives a sufficiently exact approximation for the radius of gyration of a fly wheel.

We shall now take into account the effect of the arms or spokes of the wheel.

Let n=the number of spokes, b=the area of each spoke; then we have, as before,

$$mk^2=m_1k_1^2+m_2k_2^2,$$

where, in this case, m_1=surface of the arms$=nb$, m_2=surface cir. ring$=\pi(R^2-r^2)$, m=surface whole wheel$=\pi(R^2-r^2)+nb$, k_1 =radius gyration of each spoke$=\frac{r}{\sqrt{3}}$, k_2=radius gyration of cir. ring$=\sqrt{\frac{1}{2}(R^2+r^2)}$, and k the radius of gyration of the whole fly wheel; hence we get, by substitution,

$$\{\pi(R^2-r^2)+nb\}k^2=\tfrac{1}{3}nbr^2+\tfrac{1}{2}\pi(R^4-r^4)\ldots(15),$$

whence k is readily determined.

The right hand side of this equality expresses the moment of inertia, I, of the wheel, which being substituted in eq. (6), Art. **245.**, will give the accumulated work, observing, in this case, that w expresses the weight of a unit of surface.

251. *A body, whose weight is* W, *is moved in a vertical plane, round a fixed centre* C, *by the force of gravity; to find the angular velocity, v, acquired by the body in falling from the position* CK *to* CK_1.

Let G be the centre of gravity of the body, and K its centre of gyration. Describe the arcs GG_1 and KK_1; and let fall the perpendiculars Gn and G_1m on the vertical line CD. Put $\theta=\angle$ KCD, and $\theta_1=\angle$ K_1CD; then by Art. **76.**

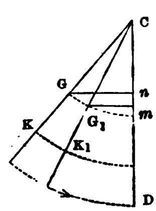


Fig. 139.

Work accumulated$=W\times nm=W\times CG(\cos\theta_1-\cos\theta)$; and by Art. **246.**, we have also

$$\text{Work accumulated}=\frac{W\times(v\,.\,CK)^2}{2g};$$

$$\therefore\ \frac{W\times(v\,.\,CK)^2}{2g}=W\times CG(\cos\theta_1-\cos\theta),$$

$$\therefore\ v^2=\frac{2g\,.\,CG}{CK^2}(\cos\theta_1-\cos\theta)\ldots(16)$$

which gives the angular velocity v, as required.

If CK falls to the vertical position CD, then $\theta_1=0$; in this case eq. (16) becomes

$$v^2=\frac{2g\,.\,CG}{CK^2}(1-\cos\theta)$$

$$=\frac{4g\,.\,CG\sin^2\frac{1}{2}\theta}{CK^2}\ldots(17).$$

If CK falls from the horizontal position to the vertical CD, then $\theta=90°$, and $\theta_1=0$; in this case eq. (16) becomes

$$v^2=\frac{2g \cdot \mathrm{CG}}{\mathrm{CK}^2} \ldots (18).$$

CHAP. XIII.

MOTION OF MACHINES WHEN THE FRICTION IS NEGLECTED.

Work accumulated in the Parts of Machines.

252. PROB.—The weights w_1, w_2, &c., forming parts of the same machine, move with the velocities v_1, v_2, &c., what must be the weight, w, which shall have the same velocity, v, as the moving power, and shall have the same accumulated work as the given weights.

Here we have, by Art. **47.**,

$$\frac{wv^2}{2g}=\frac{w_1 v_1^2}{2g}+\frac{w_2 v_2^2}{2g}+\&c.,$$

$$\therefore\ w=\frac{1}{v^2}(w_1 v_1^2+w_2 v_2^2+\&c.) \ldots (1).$$

If the velocity of the moving power varies, then since all the parts are connected by pieces of mechanism, the velocities of the weights will vary in the same ratio: thus let v become ev, then v_1 will become ev_1, and so on; in this case, therefore, we get as before

$$w=\frac{1}{(ev)^2}\{w_1(ev_1)^2+w_2(ev_2)^2+\&c.\};$$

$$\therefore\ w=\frac{1}{v^2}(w_1 v_1^2+w_2 v_2^2+\&c.);$$

which is the same result as when the initial velocity was v; thereby showing that THE WEIGHT w DOES NOT DEPEND UPON THE VELOCITY OF THE MOVING POWER.

The velocities of the weights here mentioned are, of course, the velocities of their respective centres of gyration.

253. Now, in most machines, v_1, v_2, &c., are always some constant ratio of v; putting, therefore, $v_1=e_1v$, $v_2=e_2v$, and so on, and making these substitutions in eq. (1), Art. **252.**, we get

$$w=w_1e_1^2+w_2e_2^2+\&c. \ldots (1).$$

that is to say, THE WEIGHT w IS EQUAL TO THE SUM OF THE WEIGHT OF EACH PART IN MOTION MULTIPLIED INTO THE SQUARE OF ITS VELOCITY RATIO TO THAT OF THE MOVING PRESSURE.

To express the moment of inertia of a system of bodies having different centres of rotation.

254. In addition to the notation of the preceding formulæ, let m be put for the volume of w, m_1 for that of w_1, m_2 for that of w_2, and so on; also let w' be put for the weight of a unit of volume; then $w=mw'$, $w_1=m_1w'$, $w_2=m_2w'$, and so on. Substituting these values in eq. (1), Art. **253.**, dividing by w', and multiplying by a^2, we get

$$ma^2=a^2(m_1e_1^2+m_2e_2^2+\&c.),$$

where a is put for the distance at which the moving pressure acts from the centre of motion; but $ma^2=\text{I}$, the moment of inertia of the whole system;

$$\therefore \text{I}=a^2(m_1e_1^2+m_2e_2^2+\&c.) \ldots (1).$$

Ex. The great beam of a steam engine, of uniform section, gives motion to a single crank and fly wheel; a set of wheels is connected with the axis of the fly, so as to make 10 revolutions whilst the fly makes 1. It is required to determine the weight w, which shall have the same motion as the piston with the same dynamical effect as the various parts of the engine; allowing — the length of the beam to be 20 ft., its weight 400 lbs., and its radius of gyration $\frac{10}{\sqrt{3}}$; the weight of the fly-wheel 2000 lbs., and its rad. of gy. 6 ft.; the weight of the wheels 100 lbs., and their rad. of gy. 1 ft.; and the length of the stroke of the piston 5 ft.

Taking the velocity of the piston $=5$ ft.,

$$\therefore \text{ velocity 400 lbs}=\frac{5}{10}\times\frac{10}{\sqrt{3}}=\frac{5}{\sqrt{3}};$$

Now since the fly-wheel makes one-half of a revolution, and therefore the wheels make 5 revolutions, we have

$$\begin{aligned}\text{Velocity 2000 lbs.}&=\tfrac{1}{2}\text{ circum. described}\\&=\tfrac{1}{2}\times\pi\times2\times6=6\pi,\end{aligned}$$

$$\text{Velocity } 100 \text{ lbs.} = 5 \text{ circum. described}$$
$$= 5 \times \pi \times 2 = 10\pi;$$

hence we get, by eq. (1), Art. **252.**,

$$w = \frac{1}{25}\left\{400 \times \frac{25}{3} + 2000 \times 36\pi^2 + 100 \times 100\pi^2\right\}$$

$$= \frac{400}{3} + \frac{3 \cdot 1416^2}{25}(2000 \times 36 + 10000) = 32500 \text{ lbs. nearly.}$$

Two weights P *and* W *are connected by wheel-work turning on horizontal axes; it is required to determine the motion of* P, *when the inertia of the machinery is taken into account.*

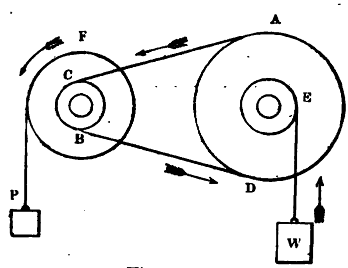


Fig. 140.

255. Let v = the velocity of P after descending the space s; e = the velocity ratio of P to W, that is, let ev = the corresponding velocity of W; w = the weight of the wheel-work calculated, Art. **252.**, to have the same velocity as P.

Now whilst P descends s ft., W ascends es ft.,

$$\therefore \text{ Work done by gravity} = P \,.\, s - W \,.\, es.$$

$$\text{Work acc. in } P = \frac{P \,.\, v^2}{2g},$$

$$\text{Work acc. in } W = \frac{W \,.\, e^2 v^2}{2g},$$

$$\text{Work acc. in the wheels} = \frac{w \,.\, v^2}{2g};$$

but the work done by gravity is accumulated in the motion given to all the parts;

$$\therefore \frac{Pv^2}{2g} + \frac{We^2v^2}{2g} + \frac{wv^2}{2g} = Ps - Wes;$$

$$\therefore v^2 = \frac{2gs(P - eW)}{P + e^2W + w} \ldots (1);$$

whence the velocity, v, of the weight P, is determined.

Here the force producing motion is obviously a uniform accelerating force; hence the relations given in Art. **27.** will apply to this case.

Now by eq. (5), Art. **27.**, we have $v^2 = 2g_1 s$;

therefore, by equality and division, we get

$$g_1=\frac{g(\text{P}-e\text{W})}{\text{P}+e^2\text{W}+w}\ldots(2),$$

which is the expression for the force accelerating P.

This value of g_1 substituted in the relation $s=t^2\times\frac{g_1}{2}$ gives

$$s=\frac{t^2}{2}\cdot\frac{g(\text{P}-e\text{W})}{\text{P}+e^2\text{W}+w}\ldots(3),$$

which expresses the relation of time and space.

We may arrive at this result without first finding the value of g_1; for

$$s=\frac{v}{2}\cdot t, \text{ or } s^2=\frac{v^2}{4}\cdot t^2.$$

Now, by substituting the value of v^2 given in eq. (1), and reducing, we readily find eq. (3).

256. If P be produced by a simple pressure, or by a pulling force, such as that produced by animal power, then the inertia of P is 0, and in this case eq. (2) becomes,

$$g_1=\frac{g(\text{P}-e\text{W})}{e^2\text{W}+w}\ldots(4).$$

257. If the wheel work be a simple pulley (such as that represented in *fig.* 12.), whose weight is w_1, radius r, and radius of gyration k; then by eq. (1), Art. **252.**,

$$w\times r^2=w_1\times k^2=w_1\times\tfrac{1}{2}r^2,$$

$$\text{and } \therefore\ w=\tfrac{1}{2}w_1;$$

moreover we have $e=1$; hence, we have, by substitution in eq. (2),

$$g_1=\frac{g(\text{P}-\text{W})}{\text{P}+\text{W}+\frac{1}{2}w}\ldots(5).$$

If w be neglected, then,

$$g_1=\frac{g(\text{P}-\text{W})}{\text{P}+\text{W}}\ldots(6).$$

258. In eq. (2), if P is simply opposed by the inertia of the machinery, then W=0, and

$$\therefore\ g_1=\frac{g\text{P}}{\text{P}+w}\ldots(7).$$

259. When the machinery turns on one axis, and W=0, then eq. (1) becomes,

$$v^2=\frac{2gs\text{P}}{\text{P}+w},$$

or putting w_1 for the actual weight of the machinery, k for its centre of gyration, and r for the radius of the wheel to which P is attached, we have, $wr^2=w_1k^2$, and $w=\frac{w_1k^2}{r^2}$; hence, by substituting in the foregoing, we get,

$$v^2=\frac{2gr^2s\text{P}}{r^2\text{P}+w_1k^2}\ldots(8).$$

Suppose the string by which P hangs to be cut, after P has descended s feet; how many revolutions would the wheel make per second.

$$\text{No. revo. per sec.}=\frac{\text{velo. circum.}}{\text{circum.}}=\frac{v}{2\pi r}$$

$$=\frac{1}{2\pi}\sqrt{\frac{2gs\text{P}}{r^2\text{P}+w_1k^2}},$$

by substituting the value of v given in eq. (8).

CHAP. XIV.

WORK OF MACHINES, THE FRICTION AND OTHER RESISTANCES BEING CONSIDERED.

MODULUS OF MACHINES, ETC., WHEN THE MOTION IS EQUABLE.

260. When all the parts of a machine attain a state of uniform motion, the work of the moving pressure is equal to the useful work done, added to the work expended on the prejudicial resistances; and, therefore, the pressures or forces, which are called into action, admit of being expressed by certain general conditions of relation. When all the resistances, which the moving pressure has to overcome, are all supposed to be referred to the moving point of the machine, they may be divided into three kinds: first, the prejudicial resistance which is simply due to the parts or pieces of the machine when unloaded, and which is therefore independent of that load; second, the resistance of the useful load itself; and third, the additional prejudicial resistance which is pro-

duced by the useful load itself, and which may be taken as a certain proportional part of the load. Thus, let P=the moving pressure; L=the useful load referred to the point where the moving pressure is applied; $f_1 \cdot L$=the resistance of friction, &c., arising from the useful load; f_2=the resistance of friction, &c., arising from the motion of the unloaded machine, so that, since f_1 and f_2 proceed from the action of the same causes, when f_1 vanishes f_2 also vanishes; then,

$$P=L+f_1L+f_2,$$

$$\text{or, } P=(1+f_1)L+f_2 \ldots (1),$$

This formula is especially applicable to the steam engine. See the Author's "Exercises in Mechanics," &c., page 41.

Now, let S=the space moved over by P; Q=the actual pressure or weight of the useful load moving over the space s; $r'=\frac{s}{S}$. the velocity ratio of P to Q; then, we have,

$$L \cdot S=Q \cdot s, \therefore L=\frac{s}{S} \cdot Q=r' \cdot Q,$$

substituting this in eq. (1), we get,

$$P=(1+f_1)r'Q+f_2 \ldots (2).$$

Again, multiplying each side of eq. (1) by S, we get,

$$P \cdot S=(1+f_1)L \cdot S+f_2 \cdot S,$$

but $P \cdot S=U$, the work of the moving pressure; $L \cdot S=U_1$, the work done upon load or the useful work of the machine;

$$\therefore U=(1+f_1)U_1+f_2 \cdot S \ldots (3),$$

which is the general equation for the work of an engine when the moving pressure and the resistances are in equilibrium.

Comparing eq. (3) with eq. (2), in order to discover the law by which the one is derived from the other, we find that THE COEFFICIENT OF U_1 IS DERIVED FROM THE COEFFICIENT OF Q BY DIVIDING r', THE VELOCITY RATIO OF THE MOVING POINT AND WORKING POINT OF THE MACHINE, AND THE LAST TERM OF EQ. (3) IS DERIVED FROM THE CORRESPONDING TERM OF EQ. (2) BY MULTIPLYING BY S, THE SPACE THROUGH WHICH THE MOVING PRESSURE IS SUPPOSED TO ACT.

261. The equality assumed in eq. (1) may be proved generally in the following manner:—

The formula of relation between P and L, in order to be general, must embrace the different states of the engine with respect to the load with which it may be supposed to work; that is to say,

the constants in the formula must hold true when $Q=0$, and also when the prejudicial resistances are equal to 0. Now when $Q=0$, the moving pressure must simply be equal to the resistance of the unloaded engine, referred to the moving point; and when the prejudicial resistances are equal to 0, the moving pressure must be equal to the useful load referred to the moving point of the engine; hence we are at liberty to assume $P=L+f_1 L+f_2$.

262. The formulæ (2) and (3) hold true when the variations of pressure are periodic. For if S and s be the spaces through which the pressures are periodic, we may assume P to be the mean pressure acting through the space S, Q the mean load acting through the space s, and f_2 the mean resistances of the unloaded engine acting through S.

263. Now let M=the modulus of the engine; then we have, by Art. **32.**,

$$M \cdot U=U_1, \text{ and } \therefore M=\frac{U_1}{U};$$

but from eq. (3), Art. **260.**,

$$U_1=\frac{U-f_2 \cdot S}{1+f_1},$$

$$\therefore M=\frac{U-f_2 \cdot S}{(1+f_1)U}=\frac{P \cdot S-f_2 \cdot S}{(1+f_1)P \cdot S}=\frac{P-f_2}{(1+f_1)P} \ldots (4);$$

which is the general expression for the modulus of an engine; where it must be observed f_1 and f_2 depend upon the nature of the prejudicial resistances of the particular machine.

Modulus of a Compound Machine.

264. Suppose a machine to be composed of n parts, which transmit work the one to the other; let $m_1, m_2, m_3, \ldots, m_n$, represent the moduli of the parts, $U_1, U_2, \ldots, U_n$, the useful work transmitted by the respective parts the one to the other, U the work at first applied, and M the modulus of the whole machine; then

$$m_1=\frac{U_1}{U}, \; m_2=\frac{U_2}{U_1}, \ldots, m_n=\frac{U_n}{U_{n-1}};$$

hence we get by multiplication,

$$m_1 \cdot m_2 \ldots, m_n=\frac{U_n}{U};$$

but $M=\frac{U_n}{U}$,

$$\therefore M=m_1 \cdot m_2 \ldots, m_n \ldots (1);$$

that is to say, THE MODULUS OF A COMPOUND MACHINE IS EQUAL TO THE CONTINUED PRODUCT OF THE MODULI OF THE PARTS.

265. Again, let $P=e_1P_1+c_1$, $P_1=e_2P_2+c_2$, $P_2=e_3P_3+c_3$, represent, according to Art. **260.**, the relations of the pressures in the consecutive parts of a machine. Eliminating the quantities P_1 and P_2, by multiplying the second equation by e_1, the third by e_1e_2, adding the resulting equations, and then striking out the terms common to both sides, we obtain

$$P=e_1e_2e_3P_3+c_1+c_2e_1+c_3e_1e_2.$$

Or generally if there are n such parts, we obviously have

$$P=e_1e_2\ldots e_nP_n+c_1+c_2e_1+c_3e_1e_2+\ \ldots\ +c_ne_1e_2\ldots e_{n-1}\ldots(1),$$

which gives the relation between the first and last pressure when the machine is bordering on motion.

Let a_1 be the velocity ratio of P to P_1, a_2 that of P_1 to P_2, &c., and a_n that of P_{n-1} to P_n; then the velocity ratio of P to P_n will be $a_1a_2\ldots a_n$; hence we have, from eqs. (2) and (3), Art. **260.**,

$$U=\frac{e_1e_2\ldots e_n}{a_1a_2\ldots a}\cdot U_n+(c_1+c_2e_1+\ \ldots\ +c_ne_1e_2\ldots e_{n-1})S\ldots(2),$$

which is the equation of useful work.

And from eq. (4), Art. **263.**, we have

$$M=\frac{a_1a_2\ldots a_n\{P-(c_1+c_2e_1+\ \ldots\ c_ne_1e_2\ldots e_{n-1})\}}{e_1e_2\ldots e_nP}\ldots(3),$$

which is the expression for the modulus.

266. If all the parts composing the machine are in all respects the same, then $e_1=e_2=\&c.=e$ (suppose); $c_1=c_2=\&c.=c$ (suppose); and $a_1=a_2=\&c.=a$; then eq. (2), Art. **265.**, becomes, putting U_1 for U_n,

$$U=\frac{e^n}{a^n}\cdot U_1+(1+e+e^2+\ \ldots\ e^{n-1})cS$$

$$=\frac{e^n}{a^n}U_1+\frac{e^n-1}{e-1}\cdot cS\ldots(1).$$

By a similar reduction eq. (3), Art. **265.**, becomes

$$M=\frac{a^n}{e^nP}\left\{P-\frac{e^n-1}{e-1}\cdot c\right\}\ldots(2).$$

MODULUS OF A MACHINE WHEN THE MOTION IS ACCELERATED OR RETARDED.

267. When the work applied at the moving point of a machine is greater than the work of the resistances, this excess of work

goes to accelerate the motion of the various parts of the machine; and then the work applied is equal to the work done upon the resistances, added to the work accumulated in the parts of the machine; on the contrary, when the work applied is less than the work of the resistances, this deficiency of work is supplied by the work accumulated in the parts of the machine; hence the motion is retarded; and then the work applied is less than the work done upon the resistances by the accumulated work lost by the various parts in motion. Thus putting u for the accumulated work gained or lost, as the case may be, by the various parts in motion, U for the work applied, and so on, as in Art. **260.**; then

Work applied=work of the resistances$\pm$acc. work gained or lost.

But by eq. (3), Art. **260**.,

$$\text{Work of the resistances}=(1+f_1)\mathrm{U}_1+f_2\mathrm{S},$$

$$\therefore\ \mathrm{U}=(1+f_1)\mathrm{U}_1+f_2\mathrm{S}+u \ldots (1).$$

Now let w be put for the weight of the whole mass in motion, referred to the moving point of the machine, as explained in Art. **252.**, v for the velocity of the moving point at the commencement, and v_1 for its velocity after it has moved over the space S; then by eq. (1), Art. **49**.,

$$u=\frac{w(v_1^2-v^2)}{2g}.$$

Making this substitution in eq. (1), we get

$$\mathrm{U}=(1+f_1)\mathrm{U}_1+f_2\mathrm{S}+\frac{w(v_1^2-v^2)}{2g} \ldots (2);$$

whence the modulus, or the relation between U and U_1, is readily determined.

In most cases of machinery, as the velocity of the machine increases, the work done upon the resistances also increases, and approaches nearer and nearer to an equality with the work applied; when this limit is attained, the work ceases to be accumulated in the parts of the machine, and it then moves with its maximum or uniform velocity. In this case, therefore, $v=v_1$, and the foregoing equality becomes the same as eq. (3), Art. **260.** Moreover, if the motion of the machine be periodical, that is to say, if it passes from v_1 to v again at stated intervals, then eq. (3), Art. **260**, will also express the relation of work between the interval of time which the machine takes in passing from the given velocity v to the same velocity again.

Velocity of a machine moving with a variable motion.
Modulus of equable motion.

268. From eq. (2), Art. **267.**, we get

$$v_1^2 = \frac{2g}{w}\{U-(1+f_1)U_1-f_2 S\}+v^2 \ldots (1),$$

where w is determined by eq. (1), Art. **253.**, viz.,

$$w = w_1 e_1^2 + w_2 e_2^2 + \&c. \ldots (2).$$

Let e be the velocity ratio of P to P_1, then $U = P \cdot S$, and $U_1 = P_1 e S$. Substituting these values in eq. (1), and reducing, we get

$$v_1^2 = \frac{2gS}{w}\{P-(1+f_1)eP_1-f_2\}+v^2 \ldots (3).$$

From this equation the velocity v_1, of the moving pressure P, is determined after passing through the space S, the velocity v at the commencement being given as well as the useful pressure P_1 and the constants of friction f_1 and f_2.

269. Now supposing the prejudicial resistances, f_1 and f_2, to remain constant, it will be observed that v_1 becomes more equable according as w, expressed in eq. (2), is increased. Hence this value of w may be regarded as the COEFFICIENT OF EQUABLE MOTION. This term was first introduced by Professor Moseley, in his work on Engineering, page 167; but the coefficient is here expressed in a form somewhat different to that given by this distinguished mathematician.

The Wheel and Axle.

270. Let P and Q be the weights hanging by cords from the wheel and axle (see *fig.* 117.), then the velocity of P to Q or r' is equal to $\frac{b}{a}$.

Comparing eq. (1), Art. **198.**, with eq. (2), Art. **260.**, we have

$$f_2 = \frac{bD + (D \pm bW) r \sin\alpha}{b(a - r \sin\alpha)} \ldots (1),$$

$$\text{and } (1+f_1)r' \text{ or } (1+f_1)\frac{b}{a} = \frac{(b+E)(b+r\sin\alpha)}{b(a-r\sin\alpha)};$$

$$\therefore\ 1+f_1 = \frac{a(b+E)(b+r\sin\alpha)}{b^2(a-r\sin\alpha)} \ldots (2).$$

These values of f_2 and $1+f_1$ substituted in eq. (3), Art. **260.**, will give the relation of work; and substituted in eq. (4), Art. **263.**, will give the modulus.

271. When the rigidity of the ropes is neglected, D and E, in the foregoing formulæ, are taken equal to 0. This case may be readily investigated without the aid of the general formula of Art. **260.**

By an easy reduction of eq. (1), Art. **191.**, we get

$$\text{F} \cdot a = \text{Q} \cdot b + (\text{P}+\text{Q}\pm\text{W})\, r \sin. \alpha \;\ldots\; (1)$$

where the sign of W is + or —, according as the pressures P and Q act downwards or upwards; and $\text{P}+\text{Q}\pm\text{W}$ is equal to the resultant, R, of all the pressures.

Now if S be the arc which the extremity of the wheel describes, that is, the space through which P moves; then $\frac{b\,\text{S}}{a}$ or s will be the arc which the axle describes, that is, the space through which Q moves; and $\frac{r\,\text{S}}{a}$ or s_1 will be the space described by the rubbing-point of the axis. Hence, multiplying eq. (1) by S, and dividing by a, we get

$$\text{P} \cdot \text{S} = \text{Q} \cdot \frac{b\,\text{S}}{a} + (\text{P}+\text{Q}\pm\text{W}) \sin \alpha \cdot \frac{r\,\text{S}}{a} \;\ldots\; (2),$$

$$= \text{Q} \cdot s + (\text{P}+\text{Q}\pm\text{W}) \sin \alpha \cdot s_1 \;\ldots\; (2),$$

$$\text{but } \text{P} \cdot \text{S} = \text{U},\ \text{Q} \cdot s = \text{U}_1,$$

$$\therefore\ \text{U} = \text{U}_1 + (\text{P}+\text{Q}\pm\text{W}) \sin \alpha \cdot s_1 \;\ldots\; (3).$$

In this equation the work of friction upon the axis is expressed by

$$\text{U}_2 = (\text{P}+\text{Q}\pm\text{W}) \sin \alpha \cdot s_1 \;\ldots\; (4).$$

Now as the angle of friction α, in eq. (2), is always very small when a metal axis turns upon a metal bearing, we have very nearly $\sin \alpha = \tan \alpha = f$; in this case eq. (3) becomes

$$\text{U} = \text{U}_1 + (\text{P}+\text{Q}\pm\text{W}) f s_1 \;\ldots\; (5).$$

Example 1. In the wheel and axle represented in *fig.* 117., let $\text{P}=800$ lbs., $a=5$, $b=1$, $r=\frac{1}{10}$, $\sin \alpha = \cdot 08$, $\therefore\ r \sin \alpha = \cdot 008$, $\text{W}=600$, circum. cord=4 in.; required the modulus of the machine.

By eqs. (1) and (2), Art. **270.**, we have

$$f_2 = \frac{1 \times 2\cdot 1 + (2\cdot 1 + 1 \times 600)\cdot 008}{5 - \cdot 008} = 1\cdot 383,$$

$$1+f_1=\frac{5(1+\cdot073)(1\times\cdot008)}{5-\cdot008}=1\cdot082;$$

Substituting these values in eq. (4), Art. **263.**, we have

$$\text{M}=\frac{800-1\cdot383}{1\cdot082\times800}=\cdot922.$$

Neglecting D in eq. (1), Art. **270.**, we find $f_2=\cdot1$ nearly, and $\text{M}=\cdot922$ as before. Thus it appears, see Art. **196.**, that for heavy weights the modulus is scarcely affected by the constant, D, of rigidity.

Example 2. Required the modulus of the machine of Example 2., Art. **199.**

Here we have found the effective pressure, Q, to be 400 lbs., when the power, P, applied is 74·35 lbs.; but we have

$$\text{M}=\frac{\text{Work of Q}}{\text{Work of P}}=\frac{\text{Q}\times a_1}{\text{P}\times a}=\frac{400\times1}{74\cdot35\times6}=\cdot9 \text{ nearly}.$$

272. When the pressures do not act parallel to each other, eq. (2), Art. **194.**, gives the approximate relation between P and Q in a state bordering on motion. See also Art. **260.** In this case the velocity ratio of P to Q is $r'=\frac{q}{p}$; hence we have by eq. (3), Art. **260.**,

$$\text{U}=\frac{p}{q}\left(\frac{q}{p}\pm\frac{r\sin\alpha}{p^2}\,.\,\text{K}\right)\text{U}_1$$

$$=\left(1\pm\frac{r\sin\alpha}{pq}\,.\,\text{K}\right)\text{U}_1\ \ldots\ (1),$$

which is the equation of useful work of the wheel and axle, when the pressures act obliquely, the weight of the wheel and axle, and the rigidity of the ropes, being neglected.

In order to introduce an allowance for the rigidity of the ropes, it is only necessary to substitute the value of Q, given in Art. **196.**, in eq. (2), Art. **194.**, and then to proceed as above.

When the weight of the wheels is taken into account.

273. In this case we have from eq. (3), Art. **195.**, by multiplying each side by $\frac{a}{a_1}$ and observing that $\frac{\text{P}\,.\,a}{\text{P}_1\,.\,a_1}=\frac{\text{U}}{\text{U}_1}$,

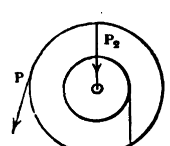


Fig. 141.

$$\text{U}=\text{U}_1\left(1+\frac{\text{J}}{aa_1}\right)\ \ldots\ (1),$$

where $J = r \sin\alpha \{a^2(1+n)^2 + a_1^2 + 2aa_1 \cos\phi_1 (1+n)\}^{\frac{1}{2}}$;

which is the equation of work, when the pressures act as in *fig.* 141., where $\cos\phi_1$ becomes minus when P acts on the same side with P_1, and in a direction contrary to it. The rigidity of the ropes may be taken into account after the manner just explained.

If P acts parallel to P_1, then $\phi_1=0$, and eq. (1) becomes

$$U = U_1 \left[1 + \frac{r \sin\alpha}{aa_1}\{a(1+n) \pm a_1\}\right] \ldots (2),$$

where the $-$ or $+$ sign of a_1 is taken according as P acts on the same side with or opposite side to P_1.

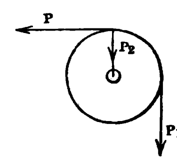


Fig. 142.

In the case of a single pulley, represented in *fig.* 142., we get from eq. (2), Art. **199**., by multiplying each side by S, substituting U for P . S, and U_1 for P_1 . S,

$$U = \left(1 + \frac{E + \sqrt{2} \,.\, r \sin\alpha}{a}\right)U_1 + \frac{S}{a}\left(D + \frac{r \sin\alpha}{\sqrt{2}} \,.\, P_2\right) \ldots (3),$$

where an allowance is made for the rigidity of the ropes.

THE WHEEL AND AXLE WHEN THE MOTION IS VARIABLE.

274. Let P and Q be weights hanging by cords from the wheel and axle (see *fig.* 70.). The general expression of work is given in eq. (2), Art. **267**. In addition to the notation of Art. **268**., let w_1 be put for the weight of the wheel, k_1 for the distance of its centre of gyration from the axis, w_2 for the weight of the axle, and k_2 for the distance of its centre of gyration from the axis; then, in order to obtain the value of the coefficient of equable motion, w, expressed by eq. (2), Art. **268**., we have

$$\text{Velo. ratio } w_1 \text{ or } e_1 = \frac{k_1}{a}; \text{ velo. ratio } w_2 \text{ or } e_2 = \frac{k_2}{a};$$

Velo. ratio $Q = \frac{b}{a}$, and velo. ratio $P = \frac{a}{a}$; hence eq. (2), Art. **268**., in this case becomes

$$w = w_1\left(\frac{k_1}{a}\right)^2 + w_2\left(\frac{k_2}{a}\right)^2 + Q\left(\frac{b}{a}\right)^2 + P\left(\frac{a}{a}\right)^2$$

$$= \frac{1}{a^2}(w_1 k_1^2 + w_2 k_2^2 + Qb^2 + Pa^2) \ldots (1),$$

which is the coefficient of equable motion. This substituted in

eq. (2), Art. **267.**, gives the equation of work; and substituted in eq. (3), Art. **268.**, we get the velocity of P; where it will be observed that the values of $1+f_1$ and f_2 are given in eqs. (1) and (2), Art. **270.**

If the wheel and axle be solid, then by Art. **246.**, $k_1=a\sqrt{\frac{1}{2}}$, $k_2=b\sqrt{\frac{1}{2}}$, and hence eq. (1) becomes

$$w=\frac{1}{a^2}(w_1\times a^2\times\tfrac{1}{2}+w_2\times b^2\times\tfrac{1}{2}+\text{Q}b^2+\text{P}a^2)$$

$$=\text{P}+\tfrac{1}{2}w_1+\left(\frac{b}{a}\right)^2(\text{Q}+\tfrac{1}{2}w_2)\ldots(2).$$

Let the wheel and axle become a single pulley, whose weight is W, then $b=a$; moreover, let the rigidity of the cord be neglected, that is, let D=0, and E=0; then we have, from eq. (2),

$$w=\text{P}+\text{Q}+\tfrac{1}{2}\text{W};$$

from eqs. (1) and (2), Art. **270.**,

$$f_2=\frac{r\sin\alpha\,.\,\text{W}}{a-r\sin\alpha},\text{ and }1+f_1=\frac{a+r\sin\alpha}{a-r\sin\alpha};$$

substituting these values in eq. (2), Art. **267.**, and (3), Art. **268.**, we get

$$\text{U}=\frac{a+r\sin\alpha}{a-r\sin\alpha}\,\text{U}_1+\frac{r\sin\alpha\,.\,\text{W}\,.\,\text{S}}{a-r\sin\alpha}+(\text{P}+\text{Q}+\tfrac{1}{2}\text{W})\,\frac{v_1^2-v^2}{2g}\ldots(3);$$

$$v_1^2=\frac{2g\text{S}}{\text{P}+\text{Q}+\frac{1}{2}\text{W}}\left\{\text{P}-\frac{a+r\sin\alpha}{a-r\sin\alpha}\,\text{Q}-\frac{r\sin\alpha\,.\,\text{W}}{a-r\sin\alpha}\right\}+v^2\ldots(4);$$

which are the equations relative to the single pulley.

If the friction of the axis be neglected, $\alpha=0$, and then eq. (4) becomes

$$v_1^2=\frac{2g\text{S}(\text{P}-\text{Q})}{\text{P}+\text{Q}+\frac{1}{2}\text{W}}+v^2\ldots(5);$$

when $v=0$, and W=0, this becomes the same as the expression given in Prob. 6., page 61.

The Pulley when the Cords are parallel to each other.

275. The wheel and axle becomes a pulley when the radius of the axle is taken equal to that of the wheel. Taking $b=a$, and therefore $r'=1$, in the different formulæ relative to the wheel and axle, we have, by eq. (2), Art. **260.**,

$$\text{P}=(1+f_1)\,\text{Q}+f_2\ldots(1);$$

and by eqs. (1) and (2), Art. **270.**, we have

$$f_2 = \frac{a\text{D} + (\text{D} \pm a\text{W})\, r \sin\alpha}{a(a - r\sin\alpha)} \ldots (2);$$

$$1 + f_1 = \frac{(a + \text{E})(a + r\sin\alpha)}{a(a - r\sin\alpha)} \ldots (3).$$

The values, given in eqs. (2) and (3), substituted in eq. (1), gives the relation of the pressures P and Q in the state bordering on motion. Moreover, these values substituted in eq. (3), Art. **260.**, gives the relation of work; and substituted in eq. (4), Art. **263.**, gives the expression for the modulus.

When the pressures act downwards, the + sign of eq. (2) is taken; but when they act upwards the − sign is taken; let c_1 be put for the value of f_2 in the former case, and c for its value in the latter case; also let e be put for the value of $1+f_1$, given in eq. (3), which, it will be observed, is independent of the directions of the cords. Making these substitutions in eq. (1), we have

$$\text{P} = e\text{Q} + c_1 \ldots (4)$$

for the state bordering on motion when the pressures act downwards; and

$$\text{P} = e\text{Q} + c \ldots (5)$$

when the pressures act upwards.

To find the modulus, &c. of a system of pulleys having one moveable pulley and one fixed pulley, such as represented in fig. 74.

276. Let Q_1 be put for the tension of the cord AR, q_1 for that of ST, and let P_1 be put for the tension of the cord DB; then we have, from eqs. (4) and (5), Art. **275.**,

$$\text{P} = e\text{Q}_1 + c_1, \text{ and } \text{Q}_1 = eq_1 + c;$$

and since the cords AR and ST together sustain the weight P_1 as well as the weight W of the pulley D,

$$\therefore\ \text{Q}_1 + q_1 = \text{P}_1 + \text{W}.$$

Eliminating Q_1 and q_1 from these three equations, we readily obtain

$$\text{P} = \frac{e^2}{1+e}\cdot\text{P}_1 + \frac{e^2\text{W} + c_1(1+e) + ec}{1+e} \ldots (1).$$

Now the velocity ratio of P and P_1 is $\frac{1}{2}$; hence we have, by the principle demonstrated in Art. 260.,

$$U=\frac{2e^2}{1+e}\cdot U_1+\frac{e^2W+c_1(1+e)+ec}{1+e}\cdot S\ldots(2),$$

which is the relation of work.

To find the modulus &c. of n *moveable pulleys.*

277. Let us now consider the case of n moveable pulleys and one fixed pulley, as represented in *fig.* 74. Let the tensions of the cords AR and ST, going over the first moveable pulley, be Q_1 and q_1; and of the cords BD and LN, going over the second moveable pulley, Q_2 and q_2; and so on; and let P_n be the weight attached to the nth moveable pulley. In order to simplify the investigation, let all the pulleys be of equal weight and size, and all the cords of the same rigidity; then the constants in the equation of pressures for all the moveable pulleys will be the same.

$$Q_1=eq_1+c, \text{ and } Q_1+q_1=Q_2+W;$$

eliminating q_1, we get

$$Q_1=\frac{e}{1+e}\cdot Q_2+\frac{eW+c}{1+e}\ldots(3);$$

or, for the sake of abbreviation,

$$Q_1=hQ_2+m.$$

Now as the successive pulleys are equal to one another, &c., we also have

$$Q_2=hQ_3+m,\ Q_3=hQ_4+m, \text{ and so on, } Q_n=hP_n+m.$$

Now from these n equations we can eliminate Q_2, Q_3, &c., in the following manner: Multiply the second by h, the third by h^2, and so on; add the resulting equations and strike out the terms common to both sides of the resulting equation; then we get

$$\begin{aligned}Q_1&=h^nP_n+m+mh+mh^2+\ldots+mh^{n-1},\\&=h^nP_n+m(1+h+h^2+\ldots+h^{n-1})\\&=h^nP_n+\frac{m(h^n-1)}{h-1};\end{aligned}$$

substituting the values of h and m, and reducing,

$$Q_1=\left(\frac{e}{1+e}\right)^nP_n+(eW+c)\left\{1-\left(\frac{e}{1+e}\right)^n\right\}$$

which applies to the moveable pulleys; but we have for the fixed pulley,

$$P = eQ_1 + c_1,$$

therefore by substitution, we get

$$P = e\left(\frac{e}{1+e}\right)^n P_n + e(eW+c)\left\{1-\left(\frac{e}{1+e}\right)^n\right\} + c_1 \ldots (4),$$

which gives the relation of the weights P and P_n, bordering on a state of motion. It will be observed that the value of e is given in expression (3), Art. **275.**, and that of c and c_1 in (2).

The velocity ratio of P and P_n is $\frac{1}{2^n}$, hence we have by the principle demonstrated in Art. 260.,

$$U = 2^n e\left(\frac{e}{1+e}\right)^n U_1 + Se(eW+c)\left\{1-\left(\frac{e}{1+e}\right)^n\right\} + Sc_1 \ldots (5);$$

which is the relation of useful work.

In a similar manner any other system of pulleys may be treated.

THE SINGLE PULLEY, WHEN THE CORDS ARE NOT PARALLEL TO EACH OTHER.

278. In this case the velocity ratio r' is unity, and hence the work is at once found by multiplying each side of eq. (3), Art. **199.**, by S, the space moved over by either of the pressures.

When a cord passes horizontally over a pulley.

279. Let W=the weight of the pulley; l=the length of the rope supported by the pulley; and w_1=the weight of each unit of the rope.

In eq. (2), Art. **271.**, make $b=a$, and put R for the sum of the pressures on the axis, then we have

$$P \cdot S = Q \cdot S + R \sin \alpha \cdot \frac{rS}{a}.$$

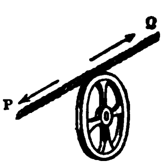


Fig. 143.

But in this case the pressure R is simply produced by the weight of the wheel added to the weight of the cord, that is, $R = W + lw_1$; moreover for the rigidity of the cord we have to substitute $\left(1+\frac{E}{a}\right)Q + \frac{D}{a}$ for Q. Making these substitutions, and observing that $P \cdot S = U$, and $Q \cdot S = U_1$, we get,

$$U=\left(1+\frac{E}{a}\right)U_1+\{(W+lw_1)r\sin\alpha+D\}\frac{S}{a}\ldots(1);$$

which is the relation of the useful work done to the work applied, taking all the resistances into account.

Dividing by S we get,

$$P=\left(1+\frac{E}{a}\right)Q+\{(W+lw)r\sin\alpha+D\}\frac{1}{a}\ldots(2),$$

what gives the relation of pressures when bordering on motion in the direction of P.

When a cord passes horizontally over n *pulleys.*

280. Let us now suppose that there are n such pulleys, sustaining each the same length of rope. Putting P_1 for Q in eq. (2), Art. **279.**, and symbolizing the coefficients as in Art. **277.**, we have

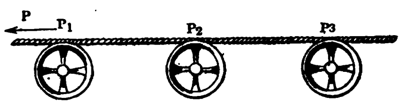


Fig. 144.

$$P=eP_1+c,\ P_1=eP_2+c,\ P_2=eP_3+c, \text{ and so on};$$

where it will be observed, that the coefficients in each equation are the same, because the pulleys are all supposed to be of the same size, &c. Hence, eq. (2), Art. **266.**, becomes, the velocity ratio, a being unity,

$$M=\frac{1}{e^nP}\left\{P-\frac{e^n-1}{e-1}c\right\}\ldots(1),$$

which is an expression for the modulus of the n pulleys.

Now $e=1+\frac{E}{a}$, and

$$e^n=\left(1+\frac{E}{a}\right)^n=1+n\cdot\frac{E}{a}+\text{ \&c.},$$

$$=1+n\cdot\frac{E}{a},$$ omitting the terms involving the powers of $\frac{E}{a}$ above the first.

Substituting in eq. (1) and reducing, we get

$$M=\frac{P-nc}{P\left(1+n\cdot\frac{E}{a}\right)}\ldots(2).$$

Again, by eq. (1), Art. **266.**, we get

$$U = e^n U_1 + \frac{e^n - 1}{e - 1} cS \ldots (3),$$

substituting the values of e and e^n as above, this equality becomes

$$U = \left(1 + \frac{nE}{a}\right) U_1 + ncS \ldots (4),$$

which is the equation of useful work.

When a cord passes horizontally and then vertically over a single pulley.

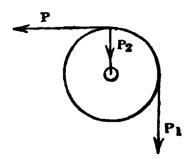


Fig. 145.

281. Multiplying each side of eq. (3), Art. **199.**, by S, observing that in this case $P \cdot S = U$, and $P_1 \cdot S = U_1$,

$$U = e_1 U_1 + c_1 S \ldots (1),$$

which is the equation of useful work, the motion being in the direction of P.

When a cord passes horizontally over any number of pulleys and then vertically over a wheel.

282. Here, supposing U_1' to be the useful work produced at the vertical rope after having left the last wheel; and U_1 the work at the horizontal rope before passing over the last wheel; then we get from eq. (1), Art. **281.**,

$$U_1 = e_1 U_1^1 + c_1 S,$$

but the relation between U, the work applied at the cord before passing over the first pulley, and U_1 is given in eq. (3), Art. **280.**; hence by eleminating U_1 between these two equations, we get

$$U = e^n e_1 U_1' + \frac{c(e^n - 1)}{e - 1} + e^n c_1 S \ldots (1),$$

which is the equation of useful work. This result may be simplified by proceeding as in Art. **277.**

Maximum Efficiency of the Wheel and Axle, and the Pulley.

283. *To determine the conditions of maximum efficiency of the wheel and axle when the forces act as represented in fig.* 120., *the rigidity of the ropes being neglected.*

The machine will obviously be most efficient when the value of U, given in eq. (1), Art. **273.**, is a minimum, the value of U_1 being constant.

There are three cases of maximum condition: first, when ϕ_1, the direction of P_1, varies, all the other elements being constant. Second, when the distance a, at which P acts from the centre of motion, varies. Third, when the weight P_2 of the machinery and a the radius of the wheel vary, this weight being assumed to be in the ratio of the square of the radius of the wheel.

CASE 1. The value of U in eq. (1), Art. 273., will be a minimum when $\cos \phi_1$ is minus, that is, when P acts on the same side with P_1, as represented in *fig.* 122.

Again, U will further be a minimum, when $\cos \phi_1$ is a maximum, that is, when $\cos \phi_1 = 1$, or when P acts parallel to P_1.

CASE 2. When a varies; a_1 P_1, P_2, $n=\frac{P_2}{P_1}$, U_1, &c. being constant.

Here U will be a minimum, when $\frac{J}{a a_1}$, or

$$\frac{r \sin \alpha}{a a_1}\{a^2(1+n)^2+a_1^2+2 a a_1 \cos \phi_1(1+n)\}^{\frac{1}{2}}, \text{ is a min.},$$

$$\text{or, } \frac{a_1^2}{a^2}+\frac{2 a_1 \cos \phi_1 (1+n)}{a}, \text{ is a min.}$$

Differentiating with respect to a, &c., see the Author's "Calculus," p. 57., we get,

$$a=-\frac{a_1}{\cos \phi_1(1+n)},$$

substituting the value of n, we get

$$a=-\frac{P_1 a_1}{\cos \phi_1 (P_1+P_2)} \dots (1).$$

Now, when P acts on the same side with P_1, and in a direction contrary to it, as in *fig.* 122., $\cos \phi_1$ becomes minus, and then this value of a becomes positive, which gives the distance at which P must act in order to work with the greatest efficiency.

If P acts parallel to P_1 and on the same side of the axis, then $\phi_1=0$, and eq. (1) becomes

$$a=\frac{P_1 a_1}{P_1+P_2} \dots (2).$$

CASE 3. When P_2 and a vary; the constants being, the velocity ratio $\frac{a}{a_1}$, P_1, U_1, &c.

Here U will be a minimum when the first expression of Case 2.

is a minimum, or, by reducing and neglecting the constant factors, when

$$\frac{1}{a^2}\left\{\frac{a^2}{a_1^2}(1+n)^2+1+2\,.\,\frac{a}{a_1}\cos\phi_1(1+n)\right\}, \text{ is a min.}$$

Let w=the weight of the wheel and axle when the radius of the wheel is unity; then $P_2=wa^2$, and $n=\frac{P_2}{P_1}=\frac{wa^2}{P_1}$. Substituting in the foregoing expression, and putting $\frac{a}{a_1}=r_1$, we get,

$$\frac{1}{a^2}\left\{r_1^2\left(1+\frac{wa^2}{P_1}\right)^2+1+2r_1\cos\phi_1\left(1+\frac{wa^2}{P_1}\right)\right\}, \text{ a min.}$$

Differentiating with respect to the variable a, putting the result equal to zero, &c., we get

$$wa^2=\frac{P_1}{r_1}(r_1^2+2r_1\cos\phi_1+1)^{\frac{1}{2}}\ldots(1),$$

which is the expression for P_2 the weight of the machinery.

Hence it follows that to attain a maximum condition of efficiency the weight P_2 of the machinery must exceed P_1 the pressure at the working point.

From eq. (1) we obtain

$$a=\sqrt{\frac{P_1}{wr_1}(r_1^2+2r_1\cos\phi_1+1)^{\frac{1}{2}}}\ldots(2),$$

which is the expression for the radius of the wheel.

If P acts horizontally $\phi_1=90$, and then eq. (1) becomes

$$P_2 \text{ or } wa^2=\frac{P_1}{r_1}(r_1^2+1)^{\frac{1}{2}}\ldots(3).$$

If P acts parallel to P_1 and on the same side of the axis, cos $\phi_1=-1$, and then eq. (1) becomes

$$P_2=\frac{P_1}{r_1}(r_1-1)\ldots(4).$$

If in eq. (1), $a_1=a$, or $r_1=1$, then eq. (1) becomes

$$P_2=P_1\sqrt{2}\times\sqrt{1+\cos\phi_1}\ldots(5)$$

which is the relation for a single pulley when P acts obliquely.

If $\phi_1=90°$, then eq. (1) becomes

$$P_2 = P_1\sqrt{2}\ldots(6),$$

which is the relation for a single pulley when P acts horizontally and P_1 vertically.

If $\phi_1=0$, then eq. (5) becomes

$$P_2 = 2P_1\ldots(7),$$

which is the relation for a single pulleys when P acts vertically.

To determine the radius of the wheel of the pulley represented in fig. 143., *to secure the condition of maximum efficiency, when the rigidity of the ropes is taken into account.*

284. Here by substituting in eq. (1) Art. **279.**, wa^2 for W, we get

$$U = \left(1+\frac{E}{a}\right)U_1 + \{(wa^2+lw_1)\,r\sin\alpha + D\}\frac{S}{a},\ \text{a min.}$$

Differentiating with respect to the variable a, substituting $Q.S$ for U_1, &c., we get

$$a^2 w = \frac{E.Q+D}{r\sin\alpha} + lw_1\ldots(1),$$

which expresses the weight of the wheel.

Hence we readily find

$$a = \sqrt{\frac{1}{w}\left(\frac{E.Q+D}{r\sin\alpha}+lw_1\right)}\ldots(2).$$

285. If the rigidity of the rope be neglected, then $E=0$, and also $D=0$; in this case, we have

$$a = \sqrt{\frac{lw_1}{w}}\ldots(3).$$

MAXIMUM VELOCITY OF AN ENGINE.

286. If an engine be moved by a pressure which increases up to a certain limit and then decreases, the maximum as well as the minimum velocity of the working point will take place when the moving pressure is equal to the sum of all the resisting pressures, for it is obvious that so long as the moving pressure continues greater than the resisting pressure so long will the motion be accelerated, and on the contrary so long as the moving pressure continues less than the resisting pressure so long will the motion

be retarded, so that the greatest velocity will take place at the moment when the moving pressure is equal to the resisting pressure, and a similar explanation will apply to the case of the minimum velocity. The following may be taken as illustrations of this beautiful dynamical law; the speed of a railway train goes on increasing until the moving pressure produced by the engine is equal to the sum of the pressures opposed to the motion; when the steam, in the cylinder of a steam engine, acts expansively, there is a certain point of the stroke of the piston where the pressure of the steam is equal to the sum of all the resistances on the piston,—at this point of the stroke the piston will have its maximum velocity; when a fly wheel is moved by a constant pressure applied to a connecting rod and crank, there are two positions of the crank corresponding to a minimum velocity, and two other positions corresponding to a maximum velocity,—and at these positions the moment of the moving pressure is equal to the moment of the resisting pressure. We shall hereafter have occasion to employ this important principle of practical mechanics.

THE FLY WHEEL.

To find the position of the crank corresponding to its maximum and minimum velocity in a single acting engine.

287. Let OP and OD be the required positions of the crank, and let us suppose P to be the constant pressure of the connecting rod acting always in a vertical line. Put Q for the *constant* resistance, acting at one foot from the axis of the fly-wheel, equivalent to the work of the engine. The motion will be accelerated from P to D, (see Art. **286.**). This acceleration will commence when the moving pressure is equal to the resisting pressure, and it will cease under the same condition. The former will correspond to the position of minimum, the latter to that of the maximum velocity. Hence at these two points the moment of P must be equal to the moment of Q, and the point D will be as much below the horizontal line OV as the point P is above it.

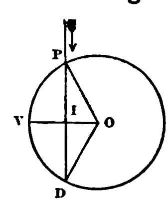


Fig. 146.

$$\therefore \text{P} \times \text{OI} = \text{Q} \times 1.$$

Again, we have by the equality of work, putting r for OP,

Work of P in 1 revo. $= 2r\text{P}$.

" Q " $= \text{Q} \times 2 \times 3{\cdot}1416$.

$$\therefore\ 2r\text{P}=\text{Q}\times 2\times 3{\cdot}1416 \ldots (1).$$

Dividing the former equation by the latter, we have

$$\frac{\text{OI}}{r}=\frac{1}{3{\cdot}1416}={\cdot}3183.$$

Now this is the cosine of the angle POV; hence from the tables of natural sines and cosines, we find POV$=71^\circ\ 27'$.

To find the dimensions of the fly wheel, such that its angular velocity may at no point differ from the mean velocity beyond a certain limit.

288. Let d and p be the maximum and minimum velocities of the wheel at the distance of one foot from the axis; W the weight of the wheel, and k the distance of the centre of gyration from the axis.

Work of P from P to D$=\text{P}\times\text{PD}=\text{P}\times 2r\sin 71^\circ\ 27'=2r\text{P}\times{\cdot}948.$
Work of the contant pressure Q from P to D

$$=\frac{\text{Q}\times 2\times 3{\cdot}1416\times 142^\circ\ 54'}{360^\circ}$$

$$=2r\text{P}\times{\cdot}3968,$$

by substituting the value given in eq. (1), Art. **287.**, and reducing.

Now by Art. **267.**, the difference of these will give us the work that goes to increase the speed of the wheel between the points P and D, that is

Work going into the wheel between P and D$=$

$$2r\text{P}\times{\cdot}948-2r\text{P}\times{\cdot}3968=r\text{P}\times 1{\cdot}1022.$$

$$\text{Accumulated work at P}=\frac{p^2k^2\text{W}}{2g}$$

$$\text{at D}=\frac{d^2k^2\text{W}}{2g}.$$

$$\therefore\ \text{Accumulated work gained from P to D}=\frac{k^2\text{W}}{2g}(d^2-p^2).$$

But this must be equal to the work before found.

$$\therefore\ \frac{k^2\text{W}}{2g}(d^2-p^2)=r\text{P}\times 1{\cdot}1022 \ldots (1).$$

Let V be the mean velocity of the wheel at one foot from the

axis, and let the extreme velocities *d* and *p*, differ from this mean velocity by an *n*th part; then

$$d=v+\frac{v}{n},\ p=v-\frac{v}{n},\ \text{and}$$

$$\therefore\ d^2-p^2=\frac{4v^2}{n}\ \ldots\ (2).$$

Let also U be the work of the engine, and N the number of double strokes performed per minute; then

$$v=2\times 3\cdot1416\times\frac{N}{60}=\cdot10472\times N\ \ldots\ (3).$$

$$U=2rPN,\ \therefore\ rP=\frac{U}{2N}\ \ldots\ (4).$$

Substituting the values given in the equations (2), (3), and (4), in eq. (1), and reducing, we get

$$W=\frac{nU}{k^2N^3}\times\frac{32\frac{1}{6}\times1\cdot1022}{4\times\cdot10472^2}$$

$$=\frac{nU}{k^2N^3}\times808\cdot2\ \ldots\ (5):$$

which is the expression for the weight of the fly wheel in lbs.

If H be put for the horse powers of engine, then $U=33{,}000\ H$; substituting this in eq. (5), and reducing to tons, we get

$$W=\frac{nH}{k^2N^3}\times11907\ \ldots\ (6)$$

which is the expression in units of tons.

Let R=the mean radius of the fly wheel, e=the depth of the rim; then by eq. (14) Art. **250.**,

$$k^2=R^2+\frac{e^2}{4};$$

substituting this in eq. (6), we get

$$W=\frac{nH}{\left(R^2+\frac{e^2}{4}\right)N^3}\times11907\ \ldots\ (7).$$

Neglecting $\frac{e^2}{4}$ as being comparatively small, we get

$$W=\frac{nH}{R^2N^3}\times11907\ \ldots\ (8),$$

where it will be observed that the weight of the wheel varies

inversely as the cube of the number of strokes performed by the engine per minute.

If a=the section of the rim in sq. feet, and 450 lbs. be taken as the weight of a cubic foot of the metal, then $W=2\pi R a \times \frac{450}{2240}$ tons, nearly.

Substituting in eq. (8), and solving the resulting equation for R, we obtain

$$R=\left\{\frac{nH}{N^3 a} \times \frac{11907 \times 2240}{2 \times 3{\cdot}1416 \times 450}\right\}^{\frac{1}{3}}$$

$$=\frac{21}{N}\sqrt[3]{\frac{nH}{a}} \dots (9),$$

which is an expression for the mean radius of a cast iron fly wheel of single acting engine, when there are given the number of strokes of the piston, the horse powers, the mean section of the rim, and the proportional variation from a mean velocity.

Proceeding in the same manner, we find, for the double acting engine, cos $\angle POV = 2 \times {\cdot}3183$, and

$$R=\frac{12}{N}\sqrt[3]{\frac{nH}{a}}$$

CHAP. XV.

CENTRIFUGAL FORCES.

289. LET a body P at the extremity of a cord CP describe a circle PEBJ about C as a centre with the uniform velocity v; to find the tension F of the cord.

If the cord were suddenly cut when the body is at any point P, it would by the first law of motion continue to move in the direction which it had at that instant; that is, it would move in the tangent PD with the uniform velocity v. The tension of the cord is the force which deflects or draws the body from the tangent line; this force is therefore called the CENTRIPETAL force. The tendency which the body has to fly off balances this centripetal force, and is therefore called the CENTRIFUGAL force.

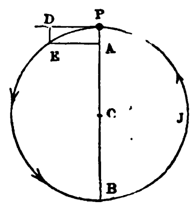


Fig. 147.

Let PE be an indefinitely small arc described by the body in the time t; PD the space which the body would describe in the same time if it were free; PDEA the parallelogram of motion; then PA will be the space due to the action of the centripetal force, or the action of the constant accelerating force which we shall call f.

By eq. (3) Art. **27.**, we have

$$AP=\tfrac{1}{2}ft^2,$$

and by the equation of uniform motion, we have

$$PD \text{ or } AE=vt;$$

by geometry, $AP=\frac{AE^2}{AB}$, or since E is indefinitely near to P, and the ultimate ratio of AB and PB is that of equality, we have, putting r for PC,

$$AP=\frac{AE^2}{AB}=\frac{AE^2}{PB}=\frac{v^2t^2}{2r},$$

therefore by equality, we have

$$\tfrac{1}{2}ft^2=\frac{v^2t^2}{2r},$$

$$\therefore f=\frac{v^2}{r} \dots (1).$$

Now if W be put for the weight of the body, we have, by eq. (1), Art. **27.**,

$$F : W :: f : g,$$

$$\text{or } F=\frac{W}{g}\cdot f;$$

and by substitution, we get

$$F=\frac{W}{g}\cdot\frac{v^2}{r} \dots (2)$$

which is the pressure tending to break the cord.

Centrifugal force of a series of bodies.

290. Let w_1, w_2, &c. be the weights of a series of bodies, in the same straight line, rotating about a common axis, r_1, r_2, &c. their respective distances from this axis; W the weight of the whole mass, r being the distance of its centre of gravity from the axis; F the centrifugal force of the whole mass, or the pressure tending to cause the whole mass to

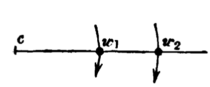


Fig. 147.

fly in a body away perpendicularly from the axis. It is required to determine an expression for F.

Let v=the velocity of a point at a unit of distance from the axis; and V=the velocity of the centre of gravity of the whole mass; then $V=rv$,

velocity of $w_1=r_1 v$,

velocity of $w_2=r_2 v$, and so on.

Now the centrifugal force of the whole mass will be equal to the sum of the centrifugal forces of the parts; hence we have, from eq. (2), Art. **289.**,

$$F=\frac{w_1}{g}\cdot\frac{(r_1 v)^2}{r_1}+\frac{w_2}{g}\cdot\frac{(r_2 v)^2}{r_2}+\&c.$$

$$=\frac{1}{g}\cdot v^2(w_1 r_1+w_2 r_2+\&c.), \text{ by Art. } \mathbf{89.},$$

$$=\frac{1}{g}\cdot v^2\cdot Wr=\frac{W}{g}\cdot\frac{v^2 r^2}{r}$$

$$=\frac{W}{g}\cdot\frac{V^2}{r}\ldots(1);$$

that is to say, THE CENTRIFUGAL FORCE OF THE SERIES OF BODIES IS EQUAL TO THE CENTRIFUGAL FORCE OF THE WHOLE MASS SUPPOSED TO BE PLACED IN THE COMMON CENTRE OF GRAVITY OF THE BODIES.

The Conical Pendulum. The Governor.

291. Let AC be the vertical axis or spindle; P the ball suspended from the rod AP; CP the perpendicular deflection of the ball from the axis; PD a vertical line representing the pressure of the weight W of the ball; PDEJ the parallelogram of pressures, of which PE will represent the centrifugal force, and PJ the strain upon the rod.

Put r=CP; l=AP; p=PJ the strain on the rod; F=PE the centrifugal force; $\theta=\angle$PAC; and v=the velocity of the ball in the circle; then from the right angled triangle PEJ, we get

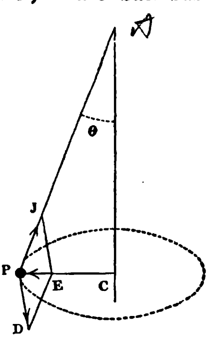


Fig. 148.

$$PJ\times\cos\theta=JE, \text{ or } p\cos\theta=W,$$

$$\therefore\ p=\frac{W}{\cos\theta}\ldots(1);$$

again, PJ $\times \sin\theta =$ PE, or $p \sin\theta =$ F;

$$\text{but by eq. (2) Art. 289., } F = \frac{W}{g} \cdot \frac{v^2}{r},$$

$$\therefore \frac{W}{g} \cdot \frac{v^2}{r} = p \sin\theta;$$

substituting the value of p given in eq. (1), and reducing, we get

$$v^2 = \frac{g r \sin\theta}{\cos\theta}$$

$$= \frac{g l \sin^2\theta}{\cos\theta} \dots (2).$$

Let t = the time of performing one revolution; then

$$t = \frac{\text{circum.}}{\text{velo.}} = \frac{2\pi r}{v}$$

$$= 2\pi \times l \sin\theta \div \sqrt{\frac{g l \sin^2\theta}{\cos\theta}}$$

$$= 2\pi \sqrt{\frac{l \cos\theta}{g}} \dots (3)$$

but $l \cos\theta =$ AC, therefore eq. (3) becomes

$$t = 2\pi \sqrt{\frac{AC}{g}} \dots (4);$$

hence it follows that the time of revolution varies as the square root of AC, whatever may be the length of the rod.

292. In the governor of a steam engine, let there be given the angle θ corresponding to a given period of rotation t, to find l the length of the rod AP.

Here from eq. (3) we readily find

$$l = \frac{t^2 g}{4\pi^2 \cos\theta} \dots (5).$$

In a given railway curve, to find the inclination of the line DE *drawn from one rail to the other, so as to secure the greatest stability from the effect of the centrifugal force.*

293. The centrifugal force will obviously have no effect in disturbing the motion of the carriage, when the resultant of the centrifugal force and the weight of the carriage, acts perpendicular to the line drawn from one rail to the other.

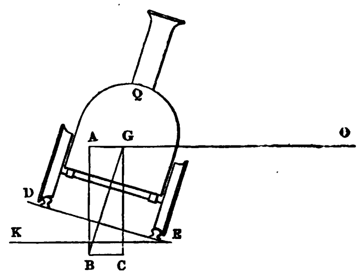


Fig. 149.

Let DEQ be a cross section of the railway carriage; G its centre of gravity; O the centre of the railway curve; EK and OG parallel to the horizon. Draw the vertical line GC, and take GC to represent the weight W of the carriage; produce OG, and take GA to represent the centrifugal force: construct the parallelogram of pressures AGCB; then the resultant GB will be perpendicular to DE, and therefore $\angle$BGC$=\angle$DEK the inclination of the line joining the rails to the horizon.

Put $\theta=\angle$DEK or $\angle$BGC; $r=$OG the radius of the railway curve; $v=$the velocity of the railway carriage in feet per second; then we have from the right angled triangle BCG,

$$\tan\theta=\frac{\text{BC}}{\text{GC}},$$

but by eq. (2), Art. **289.**, $\text{BC}=\text{AG}=\frac{\text{W}}{g}\cdot\frac{v^2}{r}$, and $\text{GC}=\text{W}$; hence we find by substitution and reduction,

$$\tan\theta=\frac{v^2}{gr}\ldots(1),$$

which gives the inclination of the line DE to the horizon.

If $h=$the rise of the outer rail above the inner one, and $b=$the breadth of the rail; then $\tan\theta=\frac{h}{b}$ nearly; and consequently eq. (1) becomes

$$\frac{h}{b}=\frac{v^2}{gr},$$

$$\therefore\ h=\frac{bv^2}{gr}\ldots(2),$$

that is to say, *the rise of the outer rail varies directly as the breadth of the rail, and inversely as the radius of the curve.*

Example. Required the rise of the outer rail, when the radius of the curve is 60 chains, the breadth of the rail 5 ft., and the speed of the carriage 30 miles per hour.

Here, $r=60\times 66$, $b=5$, and $v=\frac{30\times 5280}{60\times 60}=44$; hence we have by eq. (2),

$$h=\frac{5\times 44^2}{32\frac{1}{6}\times 60\times 66}=\cdot 076\text{ ft.}=\tfrac{9}{10}\text{ of an inch nearly.}$$

Supposing the outer and inner rails (see the last Problem) *to be on the same level, let it be required to determine the speed of the carriage which would just cause it to be overturned on* D *as a fulcrum.*

294. Let $b=$DE, the breadth of the rail; $r=$OG, the radius of the curve; $a=$the perpendicular height of the centre of gravity of the carriage above the plane of the rail; F$=$the centrifugal force, produced by the velocity v; then we have by the equality of moments,

Moment of F$=$moment of the wt. of the carriage,

$$\text{or, } \mathrm{F}\times a=\mathrm{W}\times\frac{b}{2},$$

$$\text{or, } \frac{\mathrm{W}v^2}{gr}\times a=\mathrm{W}\times\frac{b}{2},$$

$$\text{or, } \frac{av^2}{gr}=\frac{b}{2}\ \ldots\ (1),$$

$$\therefore\ v=\sqrt{\frac{bgr}{2a}}\ \ldots\ (2),$$

which gives the velocity required.

From eq. (1), it follows, that the carriage will or will not be overturned, according as

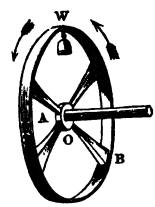
$$\frac{av^2}{gr}\text{ is greater or less than }\frac{b}{2}.$$

Example. If the width of a railway is 5 feet, the radius of the curve 600 feet, and the height of the centre of gravity of the carriage 5 feet; required the velocity of the carriage which would cause it to be overturned by the centrifugal force, the two rails being on the same level.

By eq. (2), we have, $v=\sqrt{\frac{5\times 32\frac{1}{6}\times 600}{2\times 5}}=98$ ft. per sec.; or 132 miles per hour.

PROBLEMS.

1. A body W is placed within a vertical hoop AB, which is made to revolve on a horizontal axis O; it is required to find the velocity of rotation, so that the body may not fall from the hoop.



Here the centrifugal force F, given in eq. (2), Art. **289.**, must be equal to the gravity of the body;

$$\therefore \frac{W}{g} \cdot \frac{v^2}{r} = W; \therefore v = \sqrt{gr}.$$

Fig. 150.

2. Required the same as in the last problem, supposing the hoop to revolve in a horizontal plane, and that the coefficient of friction is f.

In this case F is the pressure on the hoop, and $\therefore$ fF, the resistance of friction, must be equal to the weight of the body,

$$\therefore f\frac{Wv^2}{gr} = W, \text{ and } \therefore v = \sqrt{\frac{gr}{f}}.$$

3. In what time, t seconds, would the earth have to revolve on its axis, in order that bodies at the equator might be thrown into space, &c.?

Taking r for the radius of the earth, we have for the velocity at the equator $v = \frac{2\pi r}{t}$, equating this the result of Problem 1., and reducting, we get

$$t = 2\pi\sqrt{\frac{r}{g}} = 5075 \text{ seconds};$$

the time in which the earth actually revolves is 86160 seconds. Now by eq. (2) Art. **289.**, the centrifugal force varies directly as the square of the velocity, when the radius is constant, or, what is the same thing, inversely as the square of the time in performing a a revolution; $\therefore$ the force of gravity : the centrifugal force at the equator :: $86160^2 : 5075^2$ :: $1 : \frac{1}{290}$ nearly.

4. A weight, W, is placed on a horizontal bar, OA, which is made to revolve on a vertical axis at O, with the angular velocity v; it is required to determine the position of the weight when it is upon the point of sliding, the coefficient of friction being f.

Here, the resistance of friction must be equal to the centrifugal force.

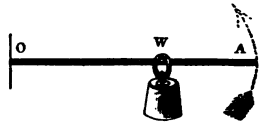


Fig. 151.

Resistance friction$=f$W,

and putting $r=$OW, we have

$$\text{the centrifugal force}=\frac{W}{g}\,.\,rv^2,$$

$$\therefore\ \frac{W}{g}\,.\,rv^2=f\text{W},$$

$$\therefore\ r=\frac{fg}{v^2}.$$

5. A weight, W, is placed within a hollow right cone, ABC, which is made to revolve upon its vertical axis AB with the given angular velocity v; it is required to determine the position of the weight when it is upon the point of being thrown out of the vessel, the coefficient of friction being f.

Draw WD perpendicular to AB; WJ perpendicular to CD; then taking DW to represent the accelerating force F, construct the parallelogram of pressures DJWE. Let $x=$WD, $\theta=\angle$ABC; then resolving the accelerating force, acting along DW, into the directions EW and JW, we have

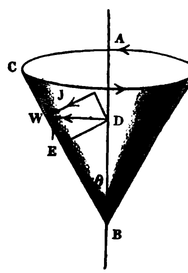


Fig. 152.

$$\text{WE}=\text{WD}\,.\,\sin\theta=\text{F}\,.\,\sin\theta,$$

$$\text{WJ}=\text{WD}\,.\,\cos\theta=\text{F}\,.\,\cos\theta.$$

By Art. **139.**,

Tendency of W down the plane CB from the force of gravity$=$W $\cos\theta$, and the perpendicular pressure of W on the plane$=$W $\sin\theta$.

Total pressure on the plane = pressure W + WJ

$$=\text{W}\sin\theta+\text{F}\cos\theta;$$

$$\therefore\ \text{Resistance friction}=f(\text{W}\sin\theta+\text{F}\cos\theta).$$

Pressure moving the body up the plane$=$WE $-$ tendency W down

$$=\text{F}\sin\theta-\text{W}\cos\theta.$$

Hence we have for the state bordering on motion;

$$\text{F}\sin\theta-\text{W}\cos\theta=f(\text{W}\sin\theta+\text{F}\cos\theta),$$

$$\therefore\ \text{F}=\frac{\text{W}(f\tan\theta+1)}{\tan\theta-f};$$

but $F=\frac{W}{g}xv^2$, hence we get by substitution and reduction

$$x=\frac{g}{v^2}\cdot\frac{f\tan\theta+1}{\tan\theta-f}\ \dots\ (1).$$

If $\theta=45°$, then $\tan\theta=1$, and we get

$$x=\frac{g}{v^2}\cdot\frac{1+f}{1-f}.$$

If $\theta=90°$, then $\tan\theta=\infty$, and eq. (1) becomes the same as that obtained in Problem 4.

If the friction be neglected, or $f=0$, then eq. (1) becomes,

$$x=\frac{g}{v^2\tan\theta}\ \dots\ (2).$$

6. If the vessel in the last problem be a hemispherical bowl whose radius is r, the *least* angular velocity requisite for maintaining the body in a given position is expressed by the equation,

$$v=\sqrt{\frac{g}{r\cos\theta}\cdot\frac{1-f\tan\theta}{f+\tan\theta}},$$

where θ is put for the angular distance of the body from the edge of the bowl.

7. If a body be sustained in a hollow vessel, as in Problem 5., to find the equation to the curve CWE, so that the tendency of the body to fly out may be the same in every position, the friction being neglected.

Let WB be the tangent, and WJ the normal to the curve. From eq. (2), Problem 5., we have

$$x\tan\theta=\frac{g}{v^2};$$

but $x=$ WD, $\tan\theta=\frac{\text{JD}}{\text{WD}}$, therefore by substitution,

$$\text{WD}\times\frac{\text{JD}}{\text{WD}}=\frac{g}{v^2},$$

$$\therefore\ \text{JD}=\frac{g}{v^2},$$

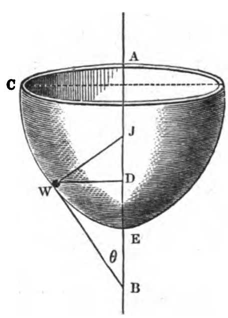


Fig. 153.

that is, the subnormal JD is a constant quantity; the curve is therefore a *parabola*, and its latus rectum; or $4a=2\text{JD}=\frac{2g}{v^2}$.

CHAP. XVI.

THE SIMPLE PENDULUM. THE CENTRES OF OSCILLATION AND PERCUSSION. THE BALLASTIC PENDULUM.

THE SIMPLE PENDULUM.

295. DEFINITION. A pendulum consists of a heavy body suspended by a thread or fine rod, and made to oscillate. When the body is supposed to be collected in a material point, and the thread by which it is suspended to be without weight, it is called a *simple pendulum*.

To find the time of an oscillation, the arc of vibration being very small.

296. Let OD be a simple pendulum oscillating in the small arc DK, and let OC be the vertical position. Let v be the velocity which the body acquires in descending the arc Db, and take bc an indefinitely small arc; then we may consider, taking the quantities at their limiting ratio, that the space bc is described with the velocity v; hence we have

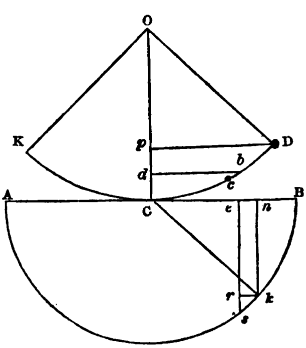


Fig. 154.

$$\text{Time through } bc = \frac{bc}{v_s} \ldots (1).$$

Draw Dp and bd perpendicular to OC; then by Art. **224.**,

$$v^2 = 2g \times pd = 2g(\text{C}p - \text{C}d) \ldots (2).$$

Now by the property of the circle KCD, we have

$$\text{C}p - \text{C}d = \frac{(\text{chord CD})^2 - (\text{chord C}b)^2}{2\text{CO}} = \frac{1}{2l}(\text{CD}^2 - \text{C}b^2),$$

very nearly, for the arc CD being small very nearly coincides with its chord.

Substituting in eq. (2), we get

$$v^2 = \frac{g}{l}(\text{CD}^2 - \text{C}b^2) \ldots (3).$$

Take CB=CD, Cn=Cb, and ne=bc; with C as a centre describe the semicircle ABs; draw nk and es perpendicular to AB, and kr parallel to it; join C and k; then by the property of the circle ABs, we have

$$\text{CD}^2 - \text{C}b^2 = \text{CB}^2 - \text{C}n^2 = nk^2.$$

Substituting in eq. (3), we get

$$v^2 = \frac{g}{l} \cdot nk^2,$$

$$\therefore\ v = nk\sqrt{\frac{g}{l}};$$

substituting in eq. (1), we find

$$\text{time through } bc = \frac{bc}{v} = \frac{rk}{v} = \frac{rk}{nk}\sqrt{\frac{l}{g}}.$$

Now the triangles Ckn and rsk are ultimately similar, and

$$\therefore\ \frac{rk}{nk} = \frac{sk}{\text{C}k} \text{ or } \frac{sk}{\text{CB}},$$

$$\therefore\ \text{time through } bc = \frac{sk}{\text{CB}}\sqrt{\frac{l}{g}}.$$

Now since this result is true for every indefinitely small portion of the arc CB, and as the sum of all the sk's is obviously equal to the semicircular arc BsA; it follows that the

$$\text{time through DCK} = \frac{\text{arc B}s\text{A}}{\text{CB}}\sqrt{\frac{l}{g}}.$$

$$\therefore\ t = \frac{\text{CB} \cdot \pi}{\text{CB}}\sqrt{\frac{l}{g}},$$

$$= \pi\sqrt{\frac{l}{g}} \ldots . (4);$$

that is, THE TIME IN PERFORMING A VIBRATION VARIES AS THE SQUARE ROOT OF THE LENGTH.

If the time be given to find the length of the pendulum, we have from eq. (4)

$$l = \frac{t^2 g}{\pi^2} \ldots (5).$$

297. Let l be the length of a pendulum which completes a vibration in t seconds, and l_1 the length of another pendulum, at the same place, which completes a vibration in t_1 seconds; then

$$t=\pi\sqrt{\frac{l}{g}},\text{ and } t_1=\pi\sqrt{\frac{l_1}{g}},$$

therefore by division and reduction, we have

$$t=t_1\sqrt{\frac{l}{l_1}},\text{ and } l=\frac{t^2}{t_1^2}\cdot l_1 \ldots (6),$$

that is, the time varies as the square root of the length, and conversely the length varies as the square of the time.

If $t_1=1$, then $l_1=39{\cdot}1393$ inches, and

$$t=\sqrt{\frac{l}{39{\cdot}1393}},\text{ and } l=39{\cdot}1393\,t^2.$$

If the time of performing a vibration, and the length of the pendulum be given, to find the accelerating force of gravity.

298. From eq. (4), we get

$$g=\frac{\pi^2 l}{t^2} \ldots (7).$$

Now it has been found by experiment that a pendulum vibrating seconds, in the latitude of London, is 39·1393 inches; hence we have

$$g=3{\cdot}1416^2\times 39{\cdot}1393=386{\cdot}3\text{ in.}=32{\cdot}19\text{ ft.}$$

which is the velocity acquired by a body descending freely by the force of gravity for one second, in the latitude of London.

To find the number of vibrations performed in a given number of seconds.

299. If n be the number of vibrations performed during N seconds, and t the time of one vibration, then

$$n=\frac{\text{N}}{t},\text{ by eq. (4),}$$

$$=\frac{\text{N}}{\pi}\sqrt{\frac{g}{l}} \ldots (8).$$

To find the number of vibrations n_1 *which a pendulum of given length,* l, *will gain in* N *seconds, by increasing the force of gravity.*

300. Let g be increased by a small quantity g_1, and n by n_1; then by eq. (8),

$$n+n_1=\frac{\text{N}}{\pi}\sqrt{\frac{g+g_1}{l}},$$

and dividing this by eq. (8), we get

$$1+\frac{n_1}{n}=\left(1+\frac{g_1}{g}\right)^{\frac{1}{2}}=1+\frac{g_1}{2g}+\&c.,$$

expanding by the binomial theorem.

Now as $\frac{g_1}{g}$ is supposed to be small, we may neglect all powers of this quantity above the first; hence we have approximately

$$\frac{n_1}{n}=\frac{g_1}{2g},$$

$$\therefore\ n_1=\frac{ng_1}{2g}\ \ldots\ (9),$$

which gives the number of vibrations gained, as required.

If r be the radius of the earth, h the height of a mountain above the level of the sea, and g_1 the quantity by which gravity is decreased at that height; then as the force of gravity varies inversely as the square of the distance from the earth's centre, we have

$$\frac{g-g_1}{g}=\frac{r^2}{(r+h)^2},$$

but by the preceding investigation $\frac{n-n_1}{n}=\left(\frac{g-g_1}{g}\right)^{\frac{1}{2}}$,

$$\therefore\ \frac{n-n_1}{n}=\frac{r}{r+h},$$

$$\therefore\ n_1=\frac{nh}{r+h},$$

$$=\frac{nh}{r}\ \ldots\ (10),\ \text{very nearly.}$$

To find the number of vibrations n_1 *which a pendulum will gain in* N *seconds by shortening the length of the pendulum.*

301. Let l be decreased by a small quantity l_1, and let n be increased by n_1; then by eq. (8), we get

$$n+n_1=\frac{\text{N}}{\pi}\sqrt{\frac{g}{l-l_1}},$$

and dividing this by eq. (8), we get

$$\frac{n+n_1}{n}=\left(\frac{l}{l-l_1}\right)=\left(1-\frac{l_1}{l}\right)^{-\frac{1}{2}}=1+\frac{l_1}{2l},\ \text{nearly;}$$

which is obtained by expanding and neglecting the terms involving the powers of $\frac{l_1}{l}$ above the first. Hence

$$n_1 = \frac{n l_1}{2l} \ldots (11)$$

which gives the expression required.

To find the length of a seconds pendulum in different latitudes.

302. It has been found that the length of a seconds pendulum at the equator is 39·0265 inches, and also that the increase of the force of gravity varies very nearly as the square of the sine of the latitude; thus if l be the length of the pendulum at the latitude L; then

$$l = 39{\cdot}0265 + {\cdot}1608 \sin^2 \text{L} \ldots (12).$$

Examples on the Simple Pendulum.

1. In what time will a pendulum vibrate, whose length is 15 inches?

Here by eq. (6) we have

$$t = \sqrt{\frac{15}{39{\cdot}1393}} = {\cdot}62 \text{ sec. nearly.}$$

2. In what time will a pendulum vibrate, whose length is double that of a seconds pendulum? *Ans.* 1·41 sec.

3. How many vibrations will a pendulum 3 feet long make in a minute? *Ans.* 62·55.

4. A pendulum which beats seconds is taken to the top of a mountain one mile high; it is required to find the number of seconds which it will lose in 12 hours, allowing the radius of the earth to be 4000 miles.

Here we have by eq. (10), $h=1$, $r=4000$,

$$\text{and } n = \frac{\text{N}}{t} = \frac{12 \times 60 \times 60}{1} = 43200,$$

$$\therefore\ n_1 = \frac{43200 \times 1}{4000} = 10{\cdot}8 \text{ sec.}$$

5. If three pendulums have their lengths as the numbers 1, 4, and 9; show that when they begin to vibrate together, they will all vibrate together again after the twelfth vibration of the shortest.

6. A pendulum which vibrates seconds at the earth's surface, is found to lose s seconds in h hours when taken to the bottom of a pit whose depth is a miles; it is required to find the radius of the earth, supposing the accelerating force of gravity to vary as the distance from the earth's centre. See Art. **300**.

Ans. Radius of the earth
$$=\cfrac{a}{1-(1-\cfrac{s}{1-\left(1-\frac{s}{3600t}\right)^2}}$$
$$=\frac{1800\,a\,t}{s} \text{ nearly.}$$

THE CENTRES OF OSCILLATION AND PERCUSSION.

303. DEFS. *The centre of oscillation* is that point in a vibrating body, where if all the matter were collected, the time of vibration would be unchanged. *The centre of percussion* is that point in a revolving body, which upon striking against an immoveable obstacle, will cause the whole of the motion of the body to be destroyed; so that if the axis were taken away, at the moment of impact, the body would have no tendency to move in any direction.

Centre of Oscillation.

304. Let C be the point of suspension of a vibrating body, G the centre of gravity, and K the centre of gyration. Suppose the body to fall from the position CO to that of CO_1; describe the arcs OO_1, GG_1, and let fall the perpendiculars $\text{O}_1 b$, $\text{O}a$, $\text{G}_1 m$, $\text{G}n$ upon the vertical line CD. Put $\theta=\angle\text{OCD}$, $\theta_1=\angle\text{O}_1\text{CD}$, and v=the angular velocity at CO_1.

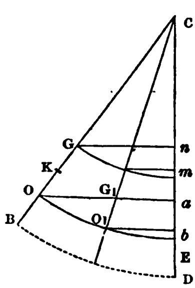


Fig. 155.

Now supposing the whole of the matter collected at O, the angular velocity of this point, acquired in falling to any position, must be the same as that which the point K really has at the same position. But the velocity acquired by O in moving from O to O_1 is that which a body would acquire in falling through ab;

$$\therefore\ (v\,.\,\text{CO})^2=2g\times ab=2g\,.\,\text{CO}\,(\cos\theta_1-\cos\theta),$$

$$\therefore\ v^2=\frac{2g}{\text{CO}}(\cos\theta_1-\cos\theta)\ \ldots\ (1).$$

But this must be the same as the value of v^2 given in eq. (18), Art. 251.,

$$\therefore \frac{2g \cdot \text{CG}}{\text{CK}^2}(\cos\theta_1 - \cos\theta) = \frac{2g}{\text{CO}}(\cos\theta_1 - \cos\theta),$$

$$\therefore \text{CG} \cdot \text{CO} = \text{CK}^2 \ldots (2);$$

that is, CK is a mean proportional between CG and CO.

For the value of CO we have

$$\text{CO} = \frac{\text{CK}^2}{\text{CG}} \ldots (3),$$

which gives the distance of the centre of oscillation from the point of suspension.

Since the body vibrates as if the whole of the mass were placed at O, the time of oscillation will be the same as a simple pendulum whose length is CO. If, therefore, l=CO, we have, by eq. (4), Art. 296., the time of a small oscillation$=\pi\sqrt{\frac{l}{g}}$.

In eq. (1) let $\theta_1=0$, then

$$v^2 = \frac{2g}{\text{CO}}(1-\cos\theta) = \frac{4g\sin^2\frac{1}{2}\theta}{\text{CO}} \ldots (4),$$

which gives the angular velocity of the pendulum at the lowest point of its vibration.

From eq. (2) we get

$$\text{CG}(\text{CG}+\text{OG}) = \text{CK}^2,$$

$$\therefore \text{CG} \cdot \text{OG} = \text{CK}^2 - \text{CG}^2;$$

but by Art. 248, CK^2-CG^2 is a constant quantity for the same body;

$$\therefore \text{CG} \cdot \text{OG} = \text{a constant}.$$

Let O be taken as the point of suspension, and let q be put for the distance of the centre of oscillation from G; then

$$\text{OG} \cdot q = \text{CG} \cdot \text{OG},$$

$$\therefore q = \text{CG};$$

that is to say, C becomes the centre of oscillation. Hence it appears, that THE CENTRES OF OSCILLATION AND SUSPENSION ARE CONVERTABLE INTO EACH OTHER.

CENTRE OF PERCUSSION.

305. Let C be the centre of rotation of a body, G its centre of

gravity, K its centre of gyration, and O its centre of percussion. Suppose the body to fall from the position CO_1 in order to acquire its angular velocity; describe the arcs OO_1 and GG_1, and let fall the perpendiculars $O_1 b$ and $G_1 n$ upon the horizontal line CO.

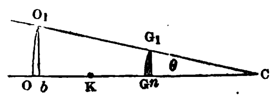


Fig. 156.

Now, if the body strike an obstacle placed at O, all the work accumulated in the body will be delivered upon that obstacle without any loss, for all the motion is supposed to be destroyed by the obstacle. Hence the work at O will be equal to the whole work accumulated in the rotating mass.

By Art. **253**,

$$\text{the weight of the mass at } O = \frac{W \cdot CK^2}{CO^2}.$$

Now, the velocity of O will be equal to that which would be acquired in falling through $O_1 b$,

$$\therefore (\text{velo. } O)^2 = 2g \times OO \sin\theta,$$

$$\therefore \text{ work accumulated at } O = \frac{\text{wt. mass at } O \times (\text{velo. } O)^2}{2g}$$

$$= \frac{1}{2g} \cdot \frac{W \cdot CK^2}{CO^2} \times 2g \times CO \sin\theta = \frac{W \cdot CK^2}{CO} \cdot \sin\theta.$$

But by Art. **111**.,

$$\text{work accumulated} = W \times G_1 n = W \times CG \sin\theta,$$

$$\therefore W \times CG \sin\theta = \frac{W \cdot CK^2}{CO} \cdot \sin\theta;$$

$$\therefore CO = \frac{CK^2}{CG};$$

hence it follows, from eq. (2), Art. **304**., that THE CENTRE OF PERCUSSION COINCIDES WITH THE CENTRE OF OSCILLATION.

306. If we conceive a force to be applied at O equal to the reaction of the obstacle, then it is obvious that this force would produce a motion in the body precisely the same as that which was destroyed; and therefore the body would rotate upon the axis C without any strain. Under these circumstances C is called THE AXIS OF SPONTANEOUS ROTATION.

The Ballistic Pendulum.

307. The Ballistic Pendulum is used for ascertaining the velo-

city of cannon balls; it consists of a heavy block of wood, suspended vertically from a horizontal axis. When the pendulum is hanging vertically and in a state of rest, a ball is fired into the block, which causes the pendulum to vibrate through an arc DB (see *fig.* 155.) This arc of vibration is known by a sharp point which, projecting from the under extremity of the block, scratches a soft composition laid in a groove formed in the arc DB. The centre of gyration is found after the method given in Art. **304.**

To find the velocity with which the ball strikes the ballistic pendulum.

308. Let W = the weight of the pendulum; P = the weight of the ball; a = the distance from C at which the ball enters the block; k = CK the radius of gyration of the pendulum with the ball; h = CG the distance of the centre of gravity of the pendulum, with the ball fixed in it, from C; l = CO, the radius of oscillation of the pendulum with the ball; $\theta = \angle$ OCD, the angle through which the pendulum is made to vibrate by the stroke of the ball; V = the velocity of the ball; v = the angular velocity of the pendulum immediately after impact; and av = the velocity of the point where the ball strikes.

By eq. (4), Art. **304.**,

$$v = 2 \sin \tfrac{1}{2}\theta \sqrt{\frac{g}{l}}.$$

Because K is the centre of gyration of the whole mass, the same angular velocity will be communicated to the pendulum as if all the mass, W + P, were collected at this point. Again, by Art. **253.**, if the mass $\frac{k^2}{a^2}(\mathrm{W}+\mathrm{P})$ be placed at E, the point where the ball strikes, the same angular velocity would be communicated to the pendulum as when the whole mass, W + P, is placed at K. Hence we may consider, that the body P impinges directly with the velocity V upon the mass $\frac{k^2}{a^2}(\mathrm{W}+\mathrm{P})$ at rest, and since the bodies are inelastic, that they move on together with the velocity av. Now, since the momentum after the stroke is equal to the momentum before it (Art. **201.**), we have

$$\mathrm{P} \times \mathrm{V} = \frac{k^2}{a^2}(\mathrm{W}+\mathrm{P}) \times av,$$

$$\therefore\ v = \frac{a\mathrm{P}\mathrm{V}}{(\mathrm{W}+\mathrm{P})k^2};$$

but by Art. **304.**, $k^2 = hl$,

$$\therefore\ v = \frac{a\text{PV}}{(\text{W}+\text{P})hl};$$

making this equal to the value of v before found, we have

$$\frac{a\text{PV}}{(\text{W}+\text{P})hl} = 2\sin\tfrac{1}{2}\theta\sqrt{\frac{g}{l}},$$

$$\therefore\ \text{V} = \frac{2h(\text{W}+\text{P})\sqrt{gl}\sin\frac{1}{2}\theta}{a\text{P}} \dots (1),$$

which gives the velocity of the ball as required.

If the pendulum makes n vibrations per minute, after being struck by the ball, then

$$\text{the time of each vibration} = \frac{60}{n};$$

therefore, by eq. (4), Art. **296.**,

$$\frac{60}{n} = \pi\sqrt{\frac{l}{g}},\ \therefore\ \sqrt{gl} = \frac{60g}{\pi n};$$

which being substituted in eq. (1), gives

$$\text{V} = \frac{120hg(\text{W}+\text{P})\sin\frac{1}{2}\theta}{\pi n a\text{P}} \dots (2).$$

PART IV.

HYDROSTATICS.

CHAP. XVII.

PRESSURE OF FLUIDS.

1. A FLUID is composed of material particles, which have a free motion amongst themselves and yield to any pressure, however small, which may be applied to them.

No fluid in nature *strictly* fulfils the definition here given.

2. The surface of still water is always horizontal.

3. Fluids transmit pressure equally in all directions.

4. The pressure of water is in proportion to its depth.

For as all the particles of a fluid press on those immediately below them, the particles at any given depth will have to sustain the weight or pressure of those which lie above them; therefore the pressure is in proportion to the depth, and this pressure acts equally in all directions,—against the side of a vessel as well as against its bottom.

Vertical Pressure on the Bottom of a Vessel.

5. This is obviously (see Art. **4.**) equal to the area of the base in feet × the depth in feet × the weight of a cubic foot of water: thus, let A = the area of the base of the vessel, a = the depth of the water, and $w = 62{\cdot}5$ lbs. the weight of a cubic foot of water; then

$$\text{Pressure on the bottom in lbs.} = A \times a \times w.$$

Examples.

(1.) Required the pressure on the bottom of a cylinder whose diameter is D, the depth of the water being a.

$$\text{Here, area base } A = D^2 \times \frac{\pi}{4};$$

$$\therefore \text{ Pressure in lbs} = \frac{1}{4}\pi D^2 aw.$$

(2.) A cubical vessel is filled with water; required the pressure upon its base.

Let a = the side of the cube; then

Area base, $A = a^2$,

$$\therefore \text{ Pressure} = a^2 \times a \times w = a^3 w.$$

TRANSMISSION OF PRESSURE BY FLUIDS.

6. This is exemplified in the Hydrostatic Press: A and a are two cylinders, containing water, connected by a pipe; P is a piston fitting the large cylinder, and p another piston, fitting the small one; now, any pressure applied to the small piston will be transmitted by the water to the large piston, so that every portion of surface in the large piston P, will be pressed upwards with the same force that an equal portion of surface in the small piston p is pressed downwards; for example, let a = the area of the piston p in square inches, A = the area of the piston P, p = the whole pressure applied to the piston p, and P = the whole pressure produced upon the piston P; then

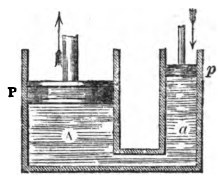


Fig. 1.

$$\frac{P}{p} = \frac{A}{a} \ldots (1),$$

which gives the equation of equilibrium.

The principle of the equality of work applies to the transmission of PRESSURES *by fluids.*

7. Let the small piston p (see *fig.* 1.) descend through the space h; then the volume of water transmitted from the small cylinder into the large one will be $a \times h$.

$$\therefore \text{ Space the large piston is raised} = \frac{a \times h}{A}.$$

From eq. (1), Art. **6.**, we get

$$p = P \times \frac{a}{A},$$

multiplying each side by h, we have

$$p \times h = P \times \frac{a \times h}{A},$$

$\therefore p \times$ space moved by $p = \text{P} \times$ space moved by P;

that is to say, the work applied to the piston p is equal to the work transmitted by the piston P.

PRESSURE ON THE SIDES OF VESSELS.

8. Let AGDQ represent the side surface of a vessel filled with water; suppose the surface to be divided into a number of indefinitely small portions, each of which may ultimately be regarded as a plane having all its points at the same distance below the upper surface of the water. Let a_1, a_2, ... be the respective areas of these portions, and h_1, h_2 ... their depths below the surface; w=the weight of a cubic foot of water; a=the area of the whole surface AGDQ; and h=the depth of its centre of gravity; then we have

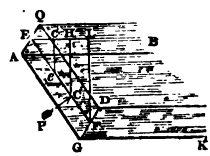


Fig. 2.

$$\text{Pressure on the surface } a_1 = a_1 h_1 w,$$

$$\text{,, \quad ,, \quad ,,} \quad a_2 = a_2 h_2 w,$$

and so on.

Now as fluids transmit pressure equally in all directions, these pressures act perpendicularly on their respective surfaces.

$\therefore$ Total pressure on the surface$= w\,(a_1 h_1 + a_2 h_2 + \ldots) \ldots (1).$

But by the property of the centre of gravity, Art. **89.**, we have

$$a \times h = a_1 h_1 + a_2 h_2 + \ldots + \&c. \ldots (2),$$

substituting in eq. (1), we get

$$\therefore \text{Total pressure on the surface} = a \times h \times w \ldots (3);$$

but $a \times h \times w$ expresses the weight of a column of water whose base is a and depth h; hence it follows, that the PRESSURE OF A FLUID UPON ANY GIVEN SURFACE IS EQUAL TO THE WEIGHT OF A COLUMN OF THAT FLUID, HAVING THE GIVEN SURFACE AS A BASE, AND THE DEPTH OF ITS CENTRE OF GRAVITY AS THE HEIGHT OF THE COLUMN.

When the side of the vessel is vertical and rectangular.

9. In this case, let a=the depth of the surface, and b=its breadth; then

$$\text{area surface} = a \times b,$$

$$\text{depth centre of gravity} = \frac{a}{2},$$

$$\therefore \text{ Pressure on the side} = a \times b \times \frac{a}{2} \times w = \frac{1}{2} a^2 b w \ldots (1).$$

When b and w are constant, *the pressure on the side varies as the square of the depth.*

Centre of Pressure.

10. Definition.—The centre of pressure is that particular point in the side of a vessel where the whole pressure upon it may be conceived to be applied without altering the effect. Thus let the surface AQDG (see *figs.* 2 and 3.) be subject to the pressure of water, then there must be a point C, in that surface where a single opposing pressure P may be applied which shall exactly balance the whole pressure of the water; this point C is called the centre of pressure.

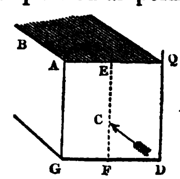


Fig. 3.

When the surface is rectangular and AQ horizontal, the centre of pressure must obviously lie in the line EF dividing the surface equally. In order to find the centre of pressure in this case let EFL (see *fig.* 2.) be a vertical plane passing through E, and let ec, CH, &c., be vertical lines drawn in this plane; then the pressure on the point e will be due to the vertical depth ec, and so on to any other points in the line EF. Now the direction of the resultant of all these vertical pressures must pass through the centre of gravity of the triangle EFL; and therefore this resultant, HC, will cut the line EF in the point C making EC equal to $\frac{2}{3}$ EF. This point, C, is obviously the centre of pressure of the surface AGDQ; hence we derive the following conclusion:—

When the side of the vessel is rectangular, the depth of the centre of pressure is equal to two-thirds the whole depth of the fluid.

Thus in *fig.* 2., we have $HC = \frac{2}{3} LF$; and in *fig.* 3., where the side is vertical, $EC = \frac{2}{3} EF$.

To determine a general expression for the depth of the centre of pressure.

11. It follows from the definition of the centre of pressure, that the sum of the moments of the surfaces a_1, a_2, &c., will be equal

to the moment of the pressure upon the whole surface regarded as acting in the centre of pressure of that surface; hence we have, taking the moments from upper surface of the fluid,

$$\text{Momt. pressure on } a_1 = a_1 h_1 w \times h_1 = a_1 h_1^2 w,$$

$$\text{,, ,, on } a_2 = a_2 h_2^2 w,$$

and so on;

Put z=the depth of the centre of pressure, then by eq. (3), Art. 8,

$$\text{Momt. whole surface} = ahw \times z,$$

$$\therefore\ ahw \times z = a_1 h^2 w + a_2 h_2^2 w + \&c.$$

$$\therefore z = \frac{a_1 h_1^2 + a_2 h_2^2 + \&c.}{ah} \ldots (1).$$

But by Art. **245.**, the numerator of this expression is the moment of inertia of the surface A D, taking the upper edge as the axis of rotation; and the denominator of the expression is the statical moment considered in relation to the same axis; hence this expression becomes

$$z = \frac{\text{moment of inertia of the surface}}{\text{statical moment of the surface}} \ldots (2).$$

It will be observed that this general formula for the depth of the centre of pressure is not restricted to any form of the surface.

Example. Let the surface AD be a rectangular plane whose length EF$=l$, and area$=d$; then (Art. **246.**) the radius of gyration of the surface is $l \times \sqrt{\frac{1}{3}}$, and the moment of inertia $a \times \frac{1}{3} l^2$; moreover the statical moment of the surface is $a \times \frac{l}{2}$; therefore by eq. (2), we have

$$z = \frac{a \times \frac{1}{3} l^2}{a \times \frac{l}{2}} = \frac{2}{3} l,$$

which result corresponds with that which has already been found.

Problems and Practical Applications.

1. To find the pressure upon the lower portion JKDG of a flood gate and also its centre of pressure.

Let C, C_1, C_2 be the centres of pressure of the surfaces AD, AK, and JD respectively. Put $a=$ AG, $a_1=$AJ, and $z=$EC_2; then by Art. 9.

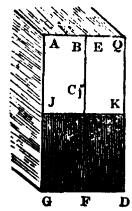


Fig. 4.

$$\text{Press. on AD}=\tfrac{1}{2}a^2bw,$$

$$\text{,, \quad ,, AK}=\tfrac{1}{2}a_1^2bw,$$

$$\therefore \text{Press. on JD}=\text{Press. on AD}-\text{Press. on AK}$$

$$=\tfrac{1}{2}bw(a^2-a_1^2)\ldots(1),$$

which is the pressure required.

To find z we have by the equality of moments,

$$\text{Press. on AK}\times\text{E}C_1+\text{Press. on JD}\times\text{E}C_2=\text{Press. on AD}\times\text{EC},$$

$$\therefore \tfrac{1}{2}a_1^2bw\times\tfrac{2}{3}a_1+\tfrac{1}{2}bw(a^2-a_1^2)\times z=\tfrac{1}{2}a^2bw\times\tfrac{2}{3}a,$$

$$\therefore z=\tfrac{2}{3}\frac{a^3-a_1^3}{a^2-a_1^2}\ldots(2),$$

which gives the depth of the centre of pressure of the rectangular portion JD.

2. When water presses upon both sides of a flood gate, to determine the pressure, &c.

Let $a=$EF the depth of the water on one side; $a_1=$$E_1$F the depth of the water on the other side; P=the effective pressure on the gate; then

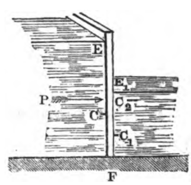


Fig. 5.

$$\text{P}=\text{Press. on EF}-\text{Press. on }E_1\text{F}$$

$$=\tfrac{1}{2}bw(a^2-a_1^2)$$

which is the pressure required.

Now let C, C_1, be the centres of pressure of the surfaces EF, and E_1F; and C_2 the point to which the resultant pressure P is applied; then by the equality of moments, we have

$$\text{P}\times\text{E}C_2=\text{Press. on EF}\times\text{EC}-\text{Press. on }E_1\text{F}\times E_1C_1;$$

proceeding as in the foregoing problem, we find

$$\text{E}C_2=\frac{2}{3}\cdot\frac{a^3-a_1^3}{a^2-a_1^2}.$$

These results correspond with those determined in the foregoing problem.

3. To find the stability of an embankment whose section has

the form of a trapezoid ACRH (see Problem 4., Art. 153.) when the water stands at the brim.

$$\text{Here } P = \tfrac{1}{2} h^2 \times 1 \times w_1,$$

where w_1 is put for the weight of a cubic foot of water; and $p =$ CP $= \frac{1}{3} h$;

∴ The moment of the water tending to overturn the embankt.

$$= P \cdot p = \tfrac{1}{2} h^2 w_1 \times \tfrac{1}{3} h = \tfrac{1}{6} h^3 w_1 \ldots (1).$$

If the embankment slopes towards the water, the foregoing expression for the moment of the water still holds true, h being its vertical depth.

This substituted in eq. (1) Prob. 4., Art. 153., will give the conditions of stability. Substituting and reducing, we get

$$h^2 = 3(a-b) \cdot \frac{w}{w_1} \left\{ \frac{2}{3}(a-b) - m \right\} + 6b \cdot \frac{w}{w_1} (a - m - \tfrac{1}{2} b) \ldots (2),$$

which gives the height of the embankment for a given modulus m.

If the embankment be upon the point of being overturned on A as a centre, then $m = 0$, and eq. (2) becomes

$$h^2 = \frac{2w}{w_1} \left\{ (a-b)^2 + 3b(a - \tfrac{1}{2} b) \right\} \ldots (3)$$

If the embankment is rectangular, then $a = b$, and eq. (2) becomes

$$h^2 = 3b \cdot \frac{w}{w_1} (b - 2m) \ldots (4).$$

From this equality, we find

$$m = \tfrac{1}{2} b - \frac{h^2 w_1}{6bw} \ldots (5).$$

Example. In eq. (5), let $h=9$, $b=5$, $w=150$ lbs., $w_1=62{\cdot}5$; required m,

$$m = 2{\cdot}5 - \frac{9^2 \times 62{\cdot}5}{6 \times 5 \times 150} = 1{\cdot}375,$$

$$\therefore \text{ The modulus of stability} = \frac{1{\cdot}375}{2{\cdot}5} = \frac{13}{25} \text{ nearly.}$$

4. The total breadth of a flood gate is $2b$ ft., and the depth a ft. The hinges are placed at e ft. from the respective ex-

tremities of the gate; required the pressure upon the lower hinge.

Here the pressure upon each half of the gate, $P = \frac{1}{2} a^2 b w$.

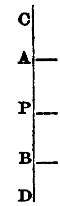


Fig. 6.

Let CD represent the height of the gate, A and B the hinges, and P the centre of pressure of the water; now since the pressure of the water at P is supported by the hinges A and B, we have by the equality of moments,

$$\text{Press. on } B \times AB = \text{Press. on } P \times AP,$$

but $AB = a - 2e$, and $AP = \frac{2}{3}a - e$,

$$\therefore \text{ Press. on } B \times (a - 2e) = \tfrac{1}{2} a^2 b w \times (\tfrac{2}{3} a - e),$$

$$\therefore \text{ Press. on } B = \frac{a^2 b w (\frac{2}{3} a - e)}{2(a - 2e)} \dots (1).$$

Example. In eq. (1), let $b = 5$, $a = 6$, $e = 1$, and $w = 62{\cdot}5$; to find the pressure on the lower hinge.

$$\text{Press. on } B = \frac{6^2 \times 5 \times 62{\cdot}5 (\frac{2}{3} \times 6 - 1)}{2(6 - 2 \times 1)} = 4218 \text{ lbs.}$$

Exercises for the Student.

1. Required the pressure on the sides of a cubical vessel filled with water, the side of the vessel being a ft. *Ans.* $125a^3$ lbs.

2. Compare the pressure on the bottom of the vessel, in the last example, with the pressure on the sides. *Ans.* The pressure on the base is one half the pressure on the sides.

3. A cylindrical vessel is filled with water. The height of the vessel is a ft., and the diameter of the base d ft.; required the pressure upon the side. *Ans.* $31\frac{1}{4} \pi a^2 d$.

4. Compare the pressure on bottom of the vessel, in the last example, with the pressure on the side. *Ans.* As $d : 2a$.

5. What must be the height of the vessel in example 3. so that the pressure on the side may be equal to the pressure on the bottom? *Ans.* The height must be one half the diameter of the base.

6. A rectangular embankment sustains the pressure of water at the brim. The height of the embankment is 6 ft., the weight of the material is 125 lbs. per cubic foot; required the breadth of the embankment so that the resultant of pressures may cut the base at the distance $\cdot 5$ ft. from the outer edge. (See eq. (4), Problem 3, Art. **11.**) *Ans.* 3 feet.

7. Required the pressure on the rectangular face AGDQ (see

fig. 2.) when it is inclined to the horizon at an angle of 45°, and AG$=2$GD$=a$. *Ans.* $\frac{62\cdot5}{4\sqrt{2}}a^3$.

8. If G, E and D, E be joined (see the last example) what will be the pressure upon the triangular portion EGD? *Ans.* $\frac{2}{3}$ the pressure on the whole surface.

Strength of Pipes and Cylindrical Steam Boilers.

12. When vessels are subject to great pressure from water or steam, the plates composing them should have an adequate thickness, in order to secure them from bursting. A cylindrical vessel may burst either longitudinally or transversely; but, as we shall hereafter demonstrate, the former takes place more frequently than the latter. In the following investigations we shall assume that when a vessel bursts, this takes place from the internal pressure tearing asunder the material of the plates. Taking this view of the subject, the force resisting the fluid pressure is the tensile force of the material or the force with which it resists the separation of its particles.

Let ABE represent the cylindrical vessel; EDCAB a longitudinal section passing through the centre; put $a=$AB the length of the cylinder, $2r=$CD the internal diameter, $e=$AC$=$DE the thickness of the plate, T$=$the tenacity in lbs. of a sq. inch of the material; then

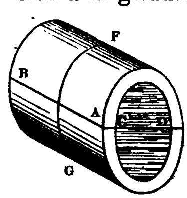


Fig. 7.

Resistance of the section ABE to rupture

$=$area section $\times$T$=2ae$T . . . (1).

Now let P$=$the pressure of the fluid in lbs. for every sq. inch. This fluid pressure, tending to produce a longitudinal rupture in ABE, will act upwards and downwards upon the internal longitudinal section, having its area$=2ar$.

$\therefore$ Pressure of the fluid to produce longitudinal rupture $=2ar$P . . . (2).

But when rupture is about to take place formula (1) must be equal to (2);

$$\therefore\ 2ae\text{T}=2ar\text{P},$$

$$\therefore\ e=\frac{r\text{P}}{\text{T}}\ldots(3),$$

which gives the thickness of the material when rupture is about to take place, under the given pressure P.

This equation shows, that for CYLINDRICAL VESSEL, COMPOSED OF THE SAME MATERIAL, THE THICKNESS SHOULD BE IN PROPORTION TO THE RADIUS.

It will readily be understood, that the longitudinal section ABE, passing through the axis of the cylinder, is the weakest longitudinal section that can be taken.

Let us now consider the conditions of rupture through the transverse section GF.

$$\text{Area section GF} = \pi(r+e)^2 - \pi r^2 = \pi e(e+2r).$$

$$\therefore \text{ Resistance section GF to rupture} = \pi e(e+2r)\mathrm{T}.$$

$$\text{Pressure fluid to produce rupture in GF} = \pi r^2 \mathrm{P}.$$

$$\therefore \ \pi e(e+2r)\mathrm{T} = \pi r^2 \mathrm{P},$$

$$\therefore \ 2e\left(\frac{e}{2r}+1\right)\mathrm{T} = r\mathrm{P},$$

neglecting $\frac{e}{2r}$ as being small, we get

$$e = \frac{r\mathrm{P}}{2\mathrm{T}} \ldots (4).$$

Comparing this with eq. (3), we find, that the transverse section has double the strength of the longitudinal section.

CHAP. XVIII.

FLOATING BODIES. SPECIFIC GRAVITY.

13. DEFINITION.—THE SPECIFIC GRAVITY of a body is its weight as compared with the weight of an equal bulk of some other body taken as a standard. For the sake of convenience pure water, at the temperature of 60°, is taken as the standard by which the specific gravities of all other substances are compared. Taking the specific gravity of water as unity, the specific gravity of any other substance is expressed by the number of times that it is heavier than an equal bulk of water: thus iron is 8 times the weight of an equal bulk of water, therefore the specific gravity of

iron is 8, and so on to other cases. Now the weight of a cubic foot of water is exactly 1000 ounces, hence we find the weight of a cubic foot of any substance by simply taking its specific gravity as so many thousands of ounces; thus the weight of a cubic foot of iron is 8000 ounces.

14. A body sinks or floats according as its specific gravity is greater or less than the fluid in which it is immersed; and when the specific gravity of the body is equal to that of the water, the body upon being immersed neither rises nor falls, but remains as it were suspended in the fluid at all depths.

When a solid body floats on a fluid, the weight of the fluid displaced is equal to the weight of the body.

15. For when a body floats it is sustained by the upward pressure of the fluid; and as there is an equilibrium of pressures, the upward pressure of the fluid must be equal to the gravity of the floating body. But this upward pressure would just support a volume of fluid equal to that which is displaced, therefore the weight of the floating body must be equal to the weight of the displaced fluid. And it is further evident that the centres of gravity of the floating body and that of the displaced fluid will be in the same vertical line.

The weight which a heavy body loses when weighed in a fluid is equal to the weight of an equal bulk of the fluid.

16. For, since the body is pressed upwards by a force equal to the weight of the displaced fluid, the weight lost by the body must be equal to this quantity.

TO FIND THE SPECIFIC GRAVITY OF A BODY.

To find the specific gravity of bodies by the hydrostatic balance.

17. The hydrostatic balance differs from an ordinary one only in having a hook attached to the under side of one of the scales. The body, whose specific gravity is to be found, is suspended from the hook by a horse-hair, and then its weight is determined. It is now weighed in water, and thus its loss of weight is ascertained. Now it has been shown that the weight which a body loses in water is equal to the weight of a portion of water equal in bulk to the

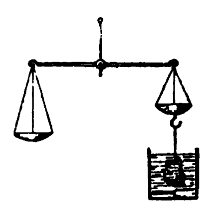


Fig. 8.

body; hence we have the following rule: *the specific gravity of a body is equal to its weight divided by the weight which it loses in water.*

Let W=the weight of the body in air, w=its weight in water, s=the specific gravity of the body, that of water being 1; then $W-w$=the weight lost, which is also the weight of the fluid displaced; therefore $W-w$ is the weight of water of the same bulk as the body whose weight is W; hence we have

Specific gravity of the body : specific gravity of water :: $W : W-w$,

$$\therefore\ s : 1 :: W : W-w,$$

$$\therefore\ s=\frac{W}{W-w}\ \ldots\ (1).$$

18. The specific gravity of liquids may be found by the hydrostatic balance, in the following manner:

Weigh a solid body in water, as well as in the liquid whose specific gravity is to be determined; then the loss in each case will be the respective weights of equal bulks of water and the liquid; therefore *the loss of weight in the liquid divided by the loss of weight in the water will give the specific gravity of the liquid.*

Let W=the weight of the body in air, w=its weight in water, and w_1=its weight in the fluid whose specific gravity s_1 is to be determined.

In this case $W-w$, and $W-w_1$ are the respective weights of equal bulks of water and the liquid;

$\therefore$ Specific gravity liquid : specific gravity water :: $W-w_1 : W-w$,

$$\therefore\ s_1 : 1 :: W-w_1 : W-w,$$

$$\therefore\ s_1=\frac{W-w_1}{W-w}\ \ldots\ (2).$$

To find the specific gravity of bodies by the hydrometer.

19. These instruments depend upon the principle, that the weight of a floating body is equal to the weight of the fluid which it displaces.

20. *Nicholson's hydrometer* is so contrived as to determine the

specific gravity of solids as well as liquids. B is a hollow ball, to which is attached a fine wire *s* supporting a dish C for receiving weights; proceeding from the under side of the ball, is the stirrup D carrying a heavy dish F for preserving the stability of the instrument when it floats, and for holding any solid body whose specific gravity is to be determined. The instrument is floated in pure water, and a weight of 1000 grains is put into the dish C; now the weight of the instrument is so adjusted that it sinks to about the middle of the fine stem; and a mark *s* is made at this point.

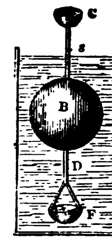


Fig. 9.

(1) To determine the specific gravity of a liquid. Place the instrument in the liquid and put weights into the dish C until the mark *s* on the stem sinks to the level of the surface of the liquid. These weights added to the weight of the instrument will be equal to the weight of the liquid displaced; but the weight of the instrument added to 1000 gr. is equal to the weight of an equal bulk of water; therefore the former sum divided by the latter will give the specific gravity of the liquid. Let the weight of the instrument $=w_1$ grains, the weight put on the dish $C=w$ grains, then we have

$$\text{Weight of displaced water} = w_1 + 1000,$$

$$\text{,, \quad ,, \quad ,, \quad liquid} = w_1 + w;$$

$$\therefore \text{ Specific gravity of liquid or } s = \frac{w_1 + w}{w_1 + 1000} \ldots (1).$$

(2) To determine the specific gravity of a solid.

Place the instrument in water, and put the solid in the upper dish C; add weights to this dish until the mark *s* on the stem sinks to a level with the fluid; then these weights together with the weight of the body must be equal to 1000 grains. Let W = the weight of the body w_2 = the weight added to the dish C; then

$$W + w_2 = 1000,$$

$$\therefore W = 1000 - w_2 \ldots (2),$$

which gives the weight of the body.

Let the body be now placed in the lower dish F, and as before, let weights be placed in the upper dish C until the mark *s* sinks to a level with the water; then these weights, together with the weight of the body *in the water*, are equal to 1000 grains. Let

w=the weight of the body in water; w_3=the weight added to the upper dish C; then

$$w+w_3=1000;$$

$$\therefore\ w=1000-w_3\ldots(3).$$

Now having found the weight of body, eq. (2), and also its weight in water, eq. (3), we have, by taking the difference of these equations,

$$\text{weight lost in water or } \mathrm{W}-w=w_3-w_2.$$

Let S be put for the specific of the body; then

$$\mathrm{S}=\frac{\text{wt. body}}{\text{wt. lost in water}}=\frac{\mathrm{W}}{\mathrm{W}-w}$$

$$=\frac{1000-w_2}{w_3-w_2}\ldots(4).$$

Example. Let $w_2=300$, $w_3=400$; then by eq. (4) we have for the specific gravity of the body

$$\mathrm{S}=\frac{1000-300}{400-300}=7.$$

21. *The common Hydrometer* has no dish at C, and the lower dish F is simply a heavy ball which serves to keep the instrument in a vertical position when floating in the liquid. There are no weights used with the instrument; but the upper stem s is graduated in such a manner as to enable the operator to ascertain the specific gravity of a liquid by the depth to which the instrument sinks in it.

Let V=the volume of the whole instrument; v=the volume included between any two consecutive graduations on the stem; n, n_1=the number of divisions on the stem above the surface when the instrument is put into fluids whose specific gravities are S and S_1 respectively; then

$$\text{vol. fluid displaced in the 1st case}=\mathrm{V}-nv,$$

$$\text{,, ,, ,, ,, the 2nd case}=\mathrm{V}-n_1v;$$

$$\therefore\ \text{wt. fluid displaced in the 1st case}=\mathrm{S}(\mathrm{V}-nv),$$

$$\text{and ,, ,, ,, ,, 2nd case}=\mathrm{S}_1(\mathrm{V}-n_1v);$$

but the weight of the fluid displaced in each case is the same, for the weight of the instrument remains the same,

$$\therefore\ \mathrm{S}(\mathrm{V}-nv)=\mathrm{S}_1(\mathrm{V}-n_1v);$$

$$\therefore \frac{s}{s_1}=\frac{V-n_1 v}{V-nv} \ldots (1);$$

which gives the ratio of the specific gravities of the two liquids.

Exercises for the Student.

In the following exercises S. G. is put for "specific gravity," and the specific gravity of water is taken equal to unity.

1. A cubic foot of water weighs 1000 oz. Find the weight of a cubical block of stone whose side is 4 ft. and S. G. 1·25.
Ans. 80,000 oz.

2. Required the number of cubic feet contained in a body whose S. G. is s, and weight w oz. *Ans.* $\frac{w}{1000 s}$.

3. If s and *s* are the S. G. of two lumps of metal whose weights are w and *w* respectively; required the S. G. of the compound metal formed by fusing the lumps together, supposing no condensation of volume to take place. *Ans.* $\frac{s\mathit{s}(w+\mathit{w})}{w\mathit{s}+\mathit{w}s}$.

4. A body, weighing 20 gr., has a S. G. of 2·5; required its weight in water. *Ans.* 12 gr.

5. A body weighs *w* grs. in water and w_1 grs. in another liquid whose S. G. is s_1; required the weight of the body.
Ans. $\frac{s_1 w - s w_1}{s_1 - s}$.

6. A Nicholson's hydrometer weighs 250 grains, and requires 726 grains to sink it to the required depth in alcohol; required the S. G. of the alcohol. *Ans.* ·781.

7. If v and *v* be put for the respective volumes of the lumps of metal in Example 3.; then it is required to find an expression for the S. G. of the compound metal. *Ans.* $\frac{v\mathit{s}+\mathit{v}s}{v+\mathit{v}}$.

8. If s_1 be put for the S. G. of the compound metal in Example 3., and w_1 for its weight; then $\frac{w_1}{s_1}=\frac{w}{s}+\frac{\mathit{w}}{\mathit{s}}$.

Floating Bodies.

22. To find the depth to which a rectangular piece of wood will sink in a fluid.

Let ABCD represent a transverse section of the body, EF being the plane of floatation. Put b=AB, a=AD, l=the length of the body, w= the weight of cubic foot of the body, and x= ED the depth of immersion, then

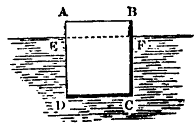


Fig. 10.

$$\text{Weight of the body} = ablw,$$

$$\text{weight of the displaced fluid} = bxl \times 62\cdot5;$$

$$\therefore\ bxl \times 62\cdot5 = ablw;$$

$$\therefore\ x = \frac{w}{62\cdot5} \times a.$$

23. A barge (supposed for the sake of simplicity to be of a rectangular shape) is l ft. long, b ft. broad, and a ft. deep, outside measure. The thickness of the planking is e ft., and the weight of a cubic foot of the timber is w lbs. To what depth, d, will the barge sink when loaded with W lbs.?

Content of the timber, V=vol. exterior solid − vol. interior

$$= abl - (a-e)(b-2e)(l-2e) \ldots (1).$$

For the sake of conciseness we shall put V for this expression, giving the volume of the timber.

Whole weight of the floating body, W^1=wt. timber+wt. load

$$= w\text{V} + \text{W}.$$

$$\text{Weight of the displaced fluid} = bdl \times 62\cdot5;$$

$$\therefore\ bdl \times 62\cdot5 = w\text{V} + \text{W};$$

$$\therefore\ d = \frac{w\text{V} + \text{W}}{62\cdot5\, bl} \ldots (2).$$

If W be required, then we get

$$\text{W} = 62\cdot5\, bdl - w\text{V} \ldots (3).$$

To determine the load necessary to sink the barge we have by making $d=a$ in eq. (3)

$$\text{W} = 62\cdot5\, abl - w\text{V} \ldots (4).$$

Obs. The irregular form of boats as they are usually constructed, renders it difficult to calculate their volume; as a tolerably near approximation to the truth, eminent surveyors take $\frac{7}{12}$ of the rectangular displacement, as determined in the foregoing

investigation, for the true displacement in boats of an ordinary curvature. Hence we have, making $V=0$ in eq. (4),

$$W=\frac{7}{12}\times 62\cdot 5\times abl\ldots(5);$$

which is the weight in lbs. requisite to sink a vessel to a given depth a, the breadth of the vessel at that depth being b, and the length l.

24. A globe of given diameter floats on water; it is required to find the weight of the displaced water, when the depth of immersion is given.

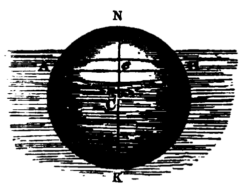


Fig. 11.

Let $r=CK=CD$ the radius of the sphere; $a=eK$ the depth of immersion; W_1=the weight of displaced water in lbs.

Here the displaced water has the form of a segment of a sphere, viz. ADK. Now we have by mensuration (see the Author's "Calculus," p. 200.)

$$\text{Content segment ADK}=\pi a^2(r-\tfrac{1}{3}a);$$

$$\therefore\ W_1=\pi a^2(r-\tfrac{1}{3}a)\times 62\cdot 5\ldots(1).$$

If the $\angle ACK=\theta$ be given; then for the value of a in eq. (1) we have

$$a=eK=CK+Ce=r(1-\cos\theta)\ldots(2).$$

To find the Buoyancy of Pontoons.

25. Pontoons are portable boats, with a covering of baulks and planks, &c., for forming floating bridges over rivers, &c. They are now usually made of tin in the shape of a cylinder with hemispherical ends.

1. Let NB represent a cylindrical pontoon with plane ends; ABD the plane of the water line parallel to the axis of the cylinder which passes through C; CN a vertical line cutting AD in e. To find the load W requisite to sink the pontoon to a given depth.

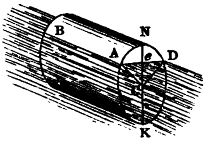


Fig. 12.

Put A=the area ADK the surface of immersion; l=AB the length of the cylinder; then we have

$$\text{vol. water displaced}=A\times l\ldots(1);$$

$$\therefore \text{ weight water displaced} = \text{A} \times l \times 62{\cdot}5;$$

$$\therefore \text{ W} = \text{A} \times l \times 62{\cdot}5 \ldots (2).$$

Where A may be found by mensuration when the radius of the circle ANDK and the depth of immersion are given. Thus

Let $r=$ CK, $\theta = \angle$ ACK; $\pi = 3{\cdot}1416+$; then

area segment ADK, or $\text{A} = r^2 \left\{ \pi \cdot \frac{\theta}{180} + \frac{1}{2} \sin 2(180-\theta) \right\} \ldots (3).$

If $\theta = 165$, or, what is the same thing, $\angle$ ACD$=30°$, then this equality becomes

$$\text{A} = r^2 \left\{ \frac{11}{12} \cdot \pi + \frac{1}{4} \right\}.$$

2. Let the pontoon be a cylinder NBK (see *fig.* 12.) with hemispherical ends. To find the load W^1 requisite to sink the pontoon to a given depth.

In this case, the solid of immersion consists of the segment ADKB of the cylinder, and two equal segments of a hemisphere, which put together give the segment of a sphere, having eK for their depth. Now eq. (2), Art. 25., gives the weight of the fluid displaced by the segment of the cylinder, and eq. (1), Art. **24.**, that of the segment of the sphere; hence we get

$$\text{W}^1 = \text{A} \times l \times 62{\cdot}5 + \pi a^2(r - \tfrac{1}{3}a) \times 62{\cdot}5$$

$$= \{\text{A}\,l + \pi a^2(r - \tfrac{1}{3}a)\} 62{\cdot}5 \ldots (4),$$

where A is given in eq. (3), Art. **25.**; and a in eq. (2), Art. **24.**

Exercises for the Student.

1. The end of a right prism is an equilateral triangle whose side is a feet; to what depth will this prism sink in the water when it floats with one of its edges undermost, its S. G. being $\frac{2}{3}$ that of water? *Ans.* $a\sqrt{\frac{1}{2}}$.

2. Required the depth of immersion, in that last example, when the prism floats with one of its edges uppermost.

Ans. $\frac{a}{2}(\sqrt{3}-1)$.

3. A rectangular barge (see Art. **23.**) l ft. long, b ft. broad, and a ft. deep, outside measure, sinks to $\frac{1}{5}$ its whole depth when unloaded; required its weight in lbs. *Ans.* $12{\cdot}5\,abl$.

4. If a rectangular barge sinks to $\frac{1}{3}$ of its whole depth when

unloaded, and to $\frac{3}{4}$ of its whole depth when loaded; required the load, the weight of the barge being w. *Ans.* $\frac{5}{4}w$.

5. The diameter of the base of a right cone is $2r$, its perpendicular height h, and its S. G. $\frac{2}{3}$ that of water; to what depth will the cone sink when it floats with its vertex undermost?

Ans. $\frac{h}{3}\sqrt[3]{18}$.

6. A hemispherical vessel, whose weight is w, floats upon a fluid with $\frac{1}{3}$ of its radius below the surface. What weight must be put into the vessel so that it may float with $\frac{2}{3}$ of its radius below the surface? *Ans.* $\frac{5}{2}w$.

7. Required the thickness of a hollow globe, made of copper, whose S. G. is 9, so that it may just float when wholly immersed in water. *Ans.* $r(1-2\sqrt[3]{\frac{1}{9}})$, *where r is the exterior radius.*

PART V.

HYDRAULICS.

CHAP. XIX.

MOTION OF FLUIDS.—EFFLUX.—RESISTANCE AND WORK OF FLUIDS.

Velocity of Water in Pipes.

26. If a fluid run through a pipe kept continually full, the velocities in the different sections vary inversely as the areas of the section.

It is here assumed that the velocity of the fluid is the same in every part of a transverse section. Let Q=the no. cubic feet of water passing through any portion of the pipe per second; v=the velocity of the water per second, in passing through that portion of the pipe whose transverse section is a; then

$$a \times v = \mathrm{Q}, \;\therefore\; v = \frac{\mathrm{Q}}{a} \ldots (1).$$

Now as Q is constant, it follows that v varies as $\frac{1}{a}$.

The quantity of water which runs through a pipe in the time t $= a \times v \times t \ldots (2)$.

Efflux of Water.

27. *The velocity of a fluid issuing from a small orifice in the bottom or side of a vessel, kept constantly full, is equal to that which a heavy body would acquire in falling through a space equal to the depth of the orifice.*

Suppose AB to be a cylinder, having a piston, P, fitting it air-

tight, ab a small portion of fluid which is being discharged from the orifice e by a pressure P applied to the piston.

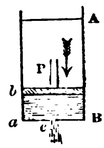


Fig. 12.

Let a = the area of the piston; s = the space through which it descends per second; W = the weight of water discharged per second; and v = the velocity of efflux.

$$\text{Work of } P = P \times s.$$

$$\text{Work accumulated in the water} = \frac{W \times v^2}{2g}.$$

Now, since ab may be taken so small that the pressure of this water may be neglected, the work due to the pressure P will be equal to the work accumulated in the water at efflux;

$$\therefore P \times s = \frac{W \times v^2}{2g};$$

but $W = a \times s \times 62 \cdot 5$,

$$\therefore P \times s = \frac{a \times s \times 62 \cdot 5 \times v^2}{2g};$$

$$\therefore \frac{v^2}{2g} = \frac{P}{62 \cdot 5 a}.$$

Now let us suppose that the pressure P is produced by a column of fluid whose height is AB; then

$$P = a \times AB \times 62 \cdot 5,$$

$$\therefore \frac{P}{62 \cdot 5 a} = AB;$$

$$\therefore \frac{v^2}{2g} = AB \ldots (1);$$

$$\text{or } v = \sqrt{2g \times AB} \ldots (2);$$

that is, the velocity of efflux is due to the vertical depth of the fluid. (See Art. **26.**, eq. (6.).)

From what has just been proved it follows that, *theoretically*, the velocity of a jet, B or C, proceeding vertically from a vessel, is such as to cause the water to rise up to the level of the water in the vessel, as shown in annexed cut.

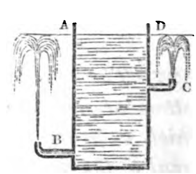


Fig. 13.

28. *To find the velocity of the issuing water, when a pressure* p *is exerted on the surface of the water.*

Let h=AB, the depth of the fluid; h_1=the height of the column of fluid which would exert the same pressure as that which is applied at the surface; then the velocity of efflux will be due to the vertical height, $h+h_1$; hence we have, by eq. (2), Art. **6.**,

$$v=\sqrt{2g(h+h_1)}\ldots(1).$$

If h_1 be taken equal to 34 feet, or the altitude of a column of water equal to the pressure of the atmosphere; then this equation becomes

$$v=\sqrt{2g(h+34)}\ldots(2).$$

This expresses the velocity with which water is projected into a vacuum, the column of fluid being h.

29. *To find the volume of water discharged from a given orifice,* bc, *made in the bottom of a vessel, when the water is kept constantly at the same level,* mn.

By Art. **27.** the theoretical velocity, v, with which the water issues, is equal to the velocity which the fluid would acquire in falling freely down from mn to bc.

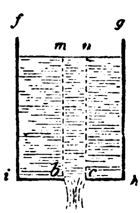


Fig. 14.

Let Q=the no. of c. ft. of water discharged in a second; a=the area of the orifice in sq. ft.; and h=the vertical height mb; then

$$v=\sqrt{2gh};$$

$$\therefore \text{ vol. water discharged per sec.}=a\sqrt{2gh};$$

$$\therefore \text{Q}=a\sqrt{2gh}\ldots(1).$$

Coefficients of Efflux and Velocity.

30. When a thin vessel empties itself in this manner, it is observed that the particles of the fluid near the top descend in vertical lines; but when they approach the bottom, they incline towards the orifice, forming curves which are convex towards the central line of the vessel; so that the issuing stream becomes more and more contracted until it reaches a point at the distance of about the radius of the orifice from the outlet, as shown in

fig. 14. The section of the stream at the point of greatest contraction was called by Newton the VENA CONTRACTA, and he found its area to be about $\frac{5}{8}$ the section of the orifice.

Let αa=the section of the vena contracta, then eq. (1) becomes

$$Q=\alpha a\sqrt{2gh} \ \ldots \ (2),$$

where α is the COEFFICIENT OF CONTRACTION.

Now this formula is only true on the supposition that the actual velocity of discharge is the same as the theoretical velocity; but this is not the case; for the actual discharge, Q_1, has been found from experiment to be less than Q as expressed by eq. (2); hence it appears that we must introduce a coefficient of velocity, in order to express the actual efflux of the fluid: let V be put for the actual velocity of efflux, and put $V=\phi v=\phi\sqrt{2gh}$, where ϕ is the COEFFICIENT OF VELOCITY. Substituting this for the velocity, we have for the actual discharge,

$$Q_1=\alpha\phi a\sqrt{2gh} \ \ldots \ (3).$$

Put $\mu=\alpha\phi$, then this equation becomes

$$Q_1=\mu a\sqrt{2gh} \ \ldots \ (4),$$

where μ is the COEFFICIENT OF EFFLUX; but $\mu=\alpha\phi$, that is to say, *the coefficient of efflux is equal to the product of the coefficient of contraction and coefficient of the velocity.*

According to the experiments of Bossut and Michelotti, $\mu=\cdot615$ or about $\frac{8}{13}$, for circular orifices of $\frac{1}{2}$ to 6 inches diameter, with from 4 to 20 feet head of water. It has, however, been shown by various experiments that the coefficient of efflux for orifices in thin plates is not exactly constant: thus it slightly decreases as we increase the orifice and the velocity of discharge, and it is considerably less for a circular orifice than it is for any irregular form.

If we take the coefficient of contraction $\alpha=\cdot64$, then we have for the coefficient of velocity,

$$\phi=\frac{\mu}{\alpha}=\frac{\cdot615}{\cdot64}=\cdot96.$$

Efflux of Water through Short Tubes, or Mouth-pieces.

31. By means of mouth-pieces the discharge of water from a given orifice is considerably increased. Thus, for a cylindrical mouth-piece, whose length does not exceed 4 times its diameter,

as in *fig.* 15., the coefficient of efflux μ is, on an average, ·813, or about $\frac{4}{5}$; for mouth-pieces having an enlargement at their exterior orifices or outlets, as well as their interior orifices, as shown in

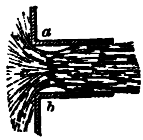


Fig. 15.

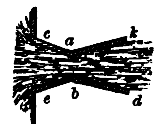


Fig. 16.

fig. 16., the coefficient of efflux μ is about 1·5526, giving a discharge greater than that which is due to the section, *a b*, of the pipe.

Now since for a simple orifice $\mu = \cdot 615$, it follows that the discharge from cylindrical tubes $= \frac{\cdot 813}{\cdot 615}$, or 1·325 times the discharge from a simple orifice in thin plates; and the discharge from the compound mouth-piece, shown in *fig.* 16. $= \frac{1\cdot 5526}{\cdot 615}$, or $2\frac{1}{2}$ times the discharge from a simple orifice; and moreover, the discharge from the compound mouth-piece $= \frac{1\cdot 5526}{\cdot 813}$, or 1·9 times the discharge from a short cylindrical tube.

In the cylindrical mouth-piece, *fig.* 15., the contraction of the fluid vein takes place at the inlet *a b* of the pipe, and not at the outlet. If a small hole were bored at *a* or *b*, no water would flow from it. As the tube is full at the point of discharge, the coefficient of velocity is the same as the coefficient of efflux.

In *fig.* 16. the mouth-piece has the form of the fluid vein; hence arises the great increase of the discharge. According to the experiments of Eytelwein, the interior diameter, *ce*, should be about 1·2 times the diameter, *ab*, of the pipe; with this enlargement the coefficient of ingress was found to be ·97; and with respect to the divergent portion, *akdb*, the angle which the sides *ak* and *bd* make with each other should be about 5° 9′, and the length of this portion should be about three times that of the other. When the length of the intervening pipe is about equal to its diameter, the coefficient of efflux is 1·35; and if the length of the intervening pipe is 60 times the diameter, this coefficient is 1·17.

Example. The head of water in a cistern is 16 feet; what quantity of water will be discharged per second from an orifice whose section contains $\frac{1}{100}$ of a sq. ft.?

Here, for a simple orifice, we have, by eq. (4), Art. **30.**,

$$Q = \cdot 615 \times \cdot 01 \sqrt{2 \times 32\tfrac{1}{6} \times 16} = \cdot 197 \text{ c. ft. nearly.}$$

And for Eytelwein's compound mouth-piece,

$$Q = \cdot 197 \times 2\tfrac{1}{2} = \cdot 49 \text{ c. ft. nearly.}$$

Coefficient of Resistance for the Ingress of Water.

32. When water flows from a cistern through pipes kept constantly full, the coefficient of ingress is the same as the coefficient of velocity.

Let W = the weight of water discharged with the actual velocity v; v_1 = the theoretical velocity of discharge due to the head of water; then

$$\text{Accumulated work lost} = (v_1^2 - v^2)\frac{W}{2g};$$

but $v = \phi\, v_1$, or $v_1 = \frac{v}{\phi}$,

$$\therefore \text{ Accumulated work lost} = \left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g} W \ldots (1);$$

therefore the head of water due to this loss of work

$$= \left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g} \ldots (2).$$

Now $\frac{1}{\phi^2} - 1$ is called the coefficient of resistance. Let $\beta = \frac{1}{\phi^2} - 1$, the coefficient of resistance, then the head of water due to the loss of accumulated work will be expressed by $\beta\frac{v^2}{2g}$.

If the pipe have the form shown in *fig.* 15., then $\phi = \cdot 813$, and $\beta = \frac{1}{\cdot 813^2} - 1 = \cdot 5$ nearly, and the loss of work $= \cdot 5\frac{v^2}{2g} W$.

If the pipe be rounded off at the part of ingress as shown in *fig.* 16., then $\phi = \cdot 97$, and $\beta = \frac{1}{\cdot 97^2} - 1 = \cdot 063$; and in this case the loss of work is $\cdot 063\frac{v^2}{2g} w$, or only about 6 per cent. In the former case the loss of work is nearly 8 times the loss in the latter case.

From the formula, $\beta = \frac{1}{\phi^2} - 1$, we get

$$\phi = \frac{1}{\sqrt{1+\beta}} \ldots (3),$$

which gives the coefficient of velocity in terms of the coefficient of resistance.

Example. If a tube 2 inches in diameter discharge ·2 c. ft. of water per second, with a head of water of 4 feet; required the coefficients of efflux and resistance, and also the loss of the head of water due to the resistance of the tube.

Here by eq. (4), Art. **30.**, we get

$$\mu \text{ or } \phi = \frac{Q}{a\sqrt{2gh}}$$

$$= \frac{\cdot 2}{\cdot 0218 \times \sqrt{2 \times 32\frac{1}{6} \times 4}} = \cdot 57,$$

which is the coefficient of efflux or velocity.

By Art. **32**.,

$$\text{the coefficient of resistance} = \frac{1}{\cdot 57^2} - 1 = 2\cdot 1 \text{ nearly.}$$

Now the velocity of efflux,

$$v = \frac{Q}{a} = \frac{\cdot 2}{\cdot 0218} = \frac{1}{\cdot 109},$$

hence by eq. (2), Art. **32**., we have

$$\text{the loss of head} = 2\cdot 1 \times \frac{1}{\cdot 109^2} \times \frac{1}{2 \times 32\frac{1}{6}} = 2\cdot 7 \text{ ft. nearly.}$$

Jets of Water.

To find the distance to which water will spout through a small orifice made in the side of a vessel.

33. Let QB be a vessel filled with water having its side AB perpendicular to the horizontal plane AH; M a small orifice made in the side of the vessel, and MH the parabolic form of the jet. On AB describe the semicircle ABD, and MN perpendicular to AB. Let V = the velocity with which the water is projected from the orifice M; and t = the time of falling down MA; then by Art. **27.**,

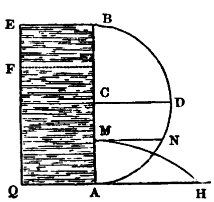


Fig. 17.

$$V = \sqrt{2g \times BM},$$

and by Art. **26.**, page 18, $t = \sqrt{\frac{2MA}{g}}$.

Now the time of a body's motion in the parabola MH is equal to the time in falling through MA.

$$\therefore \text{AH} = \text{V} \times t = \sqrt{2g \times \text{BM}} \times \sqrt{\frac{2\text{MA}}{g}}$$
$$= 2\sqrt{\text{BM} \times \text{MA}}$$
$$= 2\text{MN};$$

that is to say, *the horizontal range* AH *is equal to twice the ordinate* MN *of the semicircle.*

34. The horizontal range will obviously be a maximum when the orifice is made at C the centre of the side. In this case, the distance to which the fluid will be projected is equal to AB the height of the vessel.

To find the time that a cylindrical vessel takes to empty itself from a small orifice made in the bottom.

35. Let an orifice be made in the bottom of the vessel ABEQ (see *fig.* 17.), filled with water; and let K, k=the areas of the sections of the vessel and orifice respectively; v=the velocity of efflux from the orifice; V=the velocity of descent of the water in the vessel when full; t=the time which the vessel will take to empty itself.

Now, since the velocity of efflux varies as the square root of the depth, it follows that the velocity of descent of the water in the vessel will follow the same law; hence we conclude, Art. **26.**, that the surface of the fluid in the vessel descends according to the same law of velocity as a falling body; therefore by eq. (2), Art. **26.**, we

$$\text{EQ} = \frac{\text{V}}{2} \times t,$$

$$\text{but } \text{V} = \frac{k}{\text{K}} \cdot v = \frac{k}{\text{K}} \cdot \sqrt{2g \times \text{EQ}},$$

$$\therefore \text{EQ} = \frac{k}{2\text{K}} \sqrt{2g \times \text{EQ}} \times t,$$

$$\therefore \text{EQ} = \frac{k^2 t^2}{\text{K}^2} \cdot \tfrac{1}{2}g \ldots (1),$$

$$\therefore t = \frac{\text{K}}{k} \cdot \frac{\sqrt{\text{EQ}}}{\sqrt{\frac{1}{2}g}} \ldots (2),$$

which gives the time required.

36. To find the time t_1 required in running off the portion EBF.

Let t_1=the time in emptying AFQ; then by eq. (2)

$$t_1=\frac{K}{k}\cdot\frac{\sqrt{FQ}}{\sqrt{\frac{1}{2}g}},$$

$$\therefore\ t_1=t-t_1$$

$$=\frac{K}{k}\cdot\frac{1}{\sqrt{\frac{1}{2}g}}\{\sqrt{EQ}-\sqrt{FQ}\}\ldots(3).$$

37. To construct a CLEPSYDRA or water-clock.

In eq. (1) Art. 35, let t=12 hours=$12\times 60\times 60$ seconds, then we have

$$EQ=\frac{k^2}{K^2}\cdot\tfrac{1}{2}g(12\times 60\times 60)^2$$

which gives the depth of fluid to empty itself in 12 hours. Now if EQ be divided into 144 equal spaces, and marked upwards from the bottom of the vessel; then the marks 0, 1, 4, 9, 16, . . . , 144, will give the water level at 12, 11, 10, . . . , 2, 1, hours after the water begins to flow.

RESISTANCE AND PERCUSSION OF FLUIDS.

38. The resistance of a fluid to the motion of a body is occasioned by the force necessary to displace that fluid. Now the fluid displaced (supposing its particles to have a perfectly free motion amongst themselves) must have the same motion given to it as that of the moving body; hence the work destroyed by the fluid will be equal to the work accumulated in the fluid. Let a= the area of the front of the body presented to the fluid; v=the velocity of the body; w=the weight of a cubic foot of the fluid; R=the resistance of the fluid; then

$$\text{Weight fluid displaced per sec.}=avw,$$

but this mass has a velocity of v ft. given to it,

$$\therefore\ \text{Work expended in displacement per sec}=\frac{avw\times v^2}{2g}\ldots(1);$$

but this work is also expressed by $R\times v$,

$$\therefore\ R\times v=\frac{avw\times v^2}{2g},$$

$$\therefore\ R=\frac{aw\times v^2}{2g}\ldots(2).$$

Where we observe, that THE RESISTANCE INCREASES WITH THE SQUARE OF THE VELOCITY, AS WELL AS WITH THE EXTENT OF SURFACE PRESENTED TO THE FLUID. In extreme velocities this law does not hold strictly true. It appears also from certain recent railway experiments, that the resistance of the atmosphere, to the motion of the train, depends as well upon the length of the train, as upon the extent of the frontage of the carriages.

39. Eq. (2) will also express the force F with which the stream moving with the velocity v would strike the plane at rest; that is,

$$F=\frac{aw\times v^2}{2g}\ldots(3).$$

40. To find the force F with which a stream impinges perpendicularly on a plane which is itself in motion.

Let V=the velocity of the fluid; V_1=the velocity of the plane; a=the area of the plane; and w=the weight of a cubic foot of the fluid.

Here the relative velocity of the plane and fluid is $V-V_1$, and since this is the velocity with which every particle of the fluid strikes the plane, we have, by eq. (3), substituting $V-V_1$ for v,

$$F=\frac{aw\times(V-V_1)^2}{2g}\ldots(4).$$

WORK OF A JET OF FLUID.

41. To find the work of a jet of fluid which impinges perpendicularly upon the surface of a heavy body which is itself in motion, and whose weight is very great as compared with that of the impinging fluid.

Let W=the weight of the fluid projected on the surface per second; V=its velocity, and V_1=the velocity of the plane. Let us first suppose that the fluid after impact moves on with the body; then by eq. (7), Art. **202.**, we have

$$\text{Work lost by impact}=\frac{W(V-V_1)^2}{2g},$$

$$\therefore\ \text{Work done on the body}=\frac{W\times V^2}{2g}-\frac{W(V-V_1)^2}{2g}$$

$$=\frac{W}{2g}\{V^2-(V-V_1)^2\}\ldots(5).$$

Let AB represent a plane surface upon which the fluid impinges perpendicularly; then, if after impact the fluid and the plane move on with a common velocity, eq. (5) will express the work; but if the water leaves the plane there will be an additional portion of work lost, viz. the work remaining in the water; in this case, therefore, we have

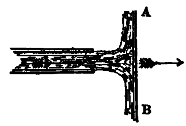


Fig. 18.

$$\text{Work done on the plane} = \frac{W}{2g}\{V^2-(V-V_1)^2\} - \frac{W}{2g}V_1^2$$

$$= \frac{W}{g}V_1(V-V_1) \ldots (6).$$

If P be put for the pressure on the plane, then the work will be equal to PV_1; making this equal to eq. (6), we get

$$P = \frac{W}{g}(V-V_1) \ldots (7).$$

If the surface is at rest or $V_1=0$, then

$$P = \frac{W}{g}V \ldots (8).$$

Let a=the section of the pipe, and h=the column of fluid equivalent to the velocity V; then $W = aV \times 62{\cdot}5$,

$$\therefore\ P = 62{\cdot}5\,a \times 2\,.\,\frac{V^2}{2g}$$

$$= 62{\cdot}5\,a \times 2h \ldots (9);$$

that is, *the pressure is equal to that which would be produced by a column of the fluid whose base is the section of the stream and perpendicular height equal to twice that which is equivalent to the velocity.*

By making the surface AB hollow, as shown in *fig.* 19, the fluid after impact leaves the plane in a direction opposite to that in which it impinges, and by this means a great saving of work is effected; for the work, remaining in the water upon leaving the curved surface, will be less than it is in the case of a plane.

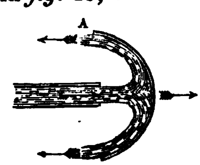


Fig. 19.

42. To find the work of a jet of fluid issuing from a nozzle with a given velocity.

Let V=the given velocity; a=the area of the nozzle, w=the wt. of a cubic foot of the fluid; here we have

$$\text{Wt. of the fluid projected per sec.} = aVw;$$

$$\therefore \text{ Work per sec.} = \frac{a v w \times v^2}{2g} = \frac{a w v^3}{2g} \ldots (1).$$

Here it will be observed, that THE WORK VARIES AS THE CUBE OF THE VELOCITY OF THE FLUID.

Let ϕ=the coefficient of velocity (see Arts. **30.** and **32.**); then we have

$$v = \phi \sqrt{2gh};$$

hence by substitution in eq. (1), we get

$$\text{Work per sec.} = \phi^3 a w h \sqrt{2gh} \ldots (2).$$

EXERCISES FOR THE STUDENT.

1. With what theoretical velocity will water issue from a small orifice $16\frac{1}{12}$ feet below the surface of the fluid? *Ans.* $32\frac{1}{6}$ ft.

2. If the area of the orifice, in the last example, is ·1 sq. ft., and the coefficient of efflux is ·615; how many cubic feet of water will be discharged per minute? *Ans.* 118·695.

3. A vessel filled with water is 4 ft. high and 1 sq. ft. in the section, and a hole of 1 sq. in. area is made in the bottom; in what time will $\frac{3}{4}$ of the water be run off, supposing the coefficient of efflux to be ·6? *Ans.* 60 sec. nearly.

Here if t be the theoretical time as derived from eq. (3) Art. **35.**; then the true time $\frac{t}{u}$.

4. What must be the form of a clepsydra, so that the water may descend through equal spaces in equal times?

CHAP. XX.

CONVEYANCE OF WATER IN PIPES, CANALS, AND RIVERS.

CONVEYANCE OF WATER IN PIPES.

43. WHEN water is conveyed from a cistern to any considerable distance in pipes, as shown in the annexed cut, the friction of the water, as it moves in the pipe, together with the obstructions presented by the bendings, &c., tends very much to retard the motion of the fluid. By the theoretical rule the velocity of discharge would be

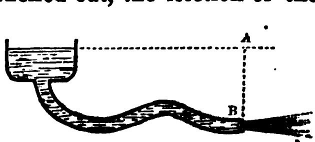


Fig. 20.

due to the vertical depth AB through which the water falls; but owing to the resistances just mentioned this is very far from being practically true; in such cases the engineer has to have recourse to some formula derived from experiment.

Horizontal Pipes.

44. Bossut found, from various experiments, that when water is conveyed from a reservoir in long horizontal pipes of the same apertures, *the discharges made in equal times are nearly inversely as the square roots of the lengths.*

FLOW OF WATER IN PIPES KEPT CONTINUALLY FULL.

When the Pipes are of the same Size throughout their Length and the Bendings are inconsiderable.

45. In this case, the resistances to the flow of water are, — the resistance due to the ingress of the water at the fountain, and the resistance of friction to the passage of the water through the pipes.

Let v = the velocity of the discharge in feet per second, d = the diameter of the pipe in feet, l = the length of the pipe in feet, h = the head or fall of water in feet, and W = the weight of water delivered per second.

It has been shown, Art. **38.**, that when a fluid in motion meets with any obstacle the resistance varies nearly with the squares of the velocity. But for a given quantity of water flowing through a pipe the resistance of friction increases with the number of points with which the water comes in contact, that is, the resistance is in proportion to the wetted surface; for every particle of water, in contact with the interior surface of the pipe, acts as a retarding force. Now let f be the resistance of friction due to a unit of diameter, length, and velocity; then the resistance in a pipe l feet long and d feet diameter with a unit of velocity will be fdl; but the quantity of water delivered by this pipe will be d^2 times that delivered by the former, therefore for the same quantity of water delivered, the resistance of friction in the latter pipe will be $\frac{fdl}{d^2}$ or $\frac{fl}{d}$, that is to say, *the resistance of friction in pipes is directly as their lengths and inversely as their diameters, the velocity being constant.* Therefore if h_1 be put for the height

of a column of water equivalent to the resistance of friction, we have

$$h_1 = e \cdot \frac{l}{d} \cdot \frac{v^2}{2g},$$

where e is a constant to be determined by experiment.

$\therefore$ Work due to the resistance of friction $= h_1 w = e\frac{l}{d} \cdot \frac{v}{2g} \cdot w$;

Work accumulated in the water at discharge $= \frac{v^2}{2g} \cdot w$.

And by eq. (1) Art. 32.,

Work due to the coefficient of velocity at ingress $= \beta\frac{v^2}{2g} \cdot w$;

Work due to the head of water $= hw$.

Now this work is equal to the work requisite in overcoming all the resistances, together with the work remaining in the water at discharge;

$$\therefore\ hw = e\frac{l}{d} \cdot \frac{v^2}{2g}w + \frac{v^2}{2g}w + \beta\frac{v^2}{2g}w,$$

$$\therefore\ h = \left(1 + \beta + e\frac{l}{d}\right)\frac{v^2}{2g} \ldots (1),$$

$$\text{and } v = \sqrt{\frac{2gdh}{el + (1+\beta)d}}$$

where the constants β and e are supposed to be determined by experiment.

The constants in eq. (1) as determined from the reduction of experiments by Poncelet, viz., $e = {\cdot}028$, and $\beta = {\cdot}5$, give us

$$v = 47{\cdot}9\sqrt{\frac{dh}{l + 54d}} \ldots (2).$$

Eytelwein gave a formula which nearly coincides with this.

46. The law of the variation of friction assumed in the foregoing investigation is not strictly true. It is more accurate to assume

$$h_1 = (e_1 v + e_2 v^2)\frac{l}{d}$$

for the column of water equivalent to the resistance of friction; where the resistance of friction depends upon the first power of

the velocity as well as upon its square. Hence, proceeding as before, we get

$$h=(e_1 v+e_2 v^2)\frac{l}{d}+\frac{v^2}{2g}+\beta\frac{v^2}{2g}\ldots(3),$$

or putting b for $\frac{1+\beta}{2g}$, and reducing, we get

$$hd=bdv^2+e_2 lv^2+e_1 lv,$$

from which equation the value of v may be found by the solution of a quadratic; but the following method of approximating to the value of v will be found more convenient. By reduction, we get

$$v\left(1+\frac{e_1 l}{bd+e_2 l}\cdot\frac{1}{v}\right)^{\frac{1}{2}}=\sqrt{\frac{hd}{bd+e_2 l}}.$$

Expanding by the binomial theorem

$$\left(1+\frac{e_1 l}{bd+e_2 l}\cdot\frac{1}{v}\right)^{\frac{1}{2}}=1+\frac{e_1 l}{2(bd+e_2 l)}\cdot\frac{1}{v}+\&c.$$

Now as e_2 is found to be considerably greater than e_1, we may neglect all the terms of this expansion after the second; therefore by substitution, we get

$$v\left(1+\frac{e_1 l}{2(bd+e_2 l)}\cdot\frac{1}{v}\right)=\sqrt{\frac{hd}{bd+e_2 l}}.$$

$$\therefore\ v=\sqrt{\frac{hd}{bd+e_2 l}}-\frac{e_1 l}{2(bd+e_2 l)}\ldots(4).$$

We have now to assign the values of the constants in this expression. From Art. 32., $\beta=\cdot5$, $\therefore\ b=\frac{1+\beta}{2g}=\frac{1\cdot5}{64\frac{1}{3}}=\cdot0234$, and taking the mean of the reductions of Prony and d'Aubuisson, we find $e_1=\cdot00007$, and $e_2=\cdot00042$. Substituting these values in eq. (4), and reducing, we get

$$v=\sqrt{\frac{2380hd}{l+54d}}-\frac{1}{12}\cdot\frac{l}{l+54d}\ldots(5).$$

If the last term of this formula be neglected, which may be done when h is not very small, we get

$$v=\sqrt{\frac{2380hd}{l+54d}}\ldots(6).$$

which is very nearly the same as formula (2).

When the pipes are very long, or $\frac{d}{l}$ is a small quantity, eqs. (2) and (5) become

$$v=47{\cdot}9\sqrt{\frac{hd}{l}}\ldots(7),$$

$$\text{and, } v=\sqrt{\frac{2380hd}{l}-\frac{1}{12}}\ldots(8).$$

Example 1. The length of a water-pipe is 5780 feet, the head of water 170 feet, and the diameter of the pipe 6 inches or ·5 ft.; required the velocity of discharge.

By formula (3)

$$v=47{\cdot}9\sqrt{\frac{{\cdot}5\times170}{5780+54\times{\cdot}5}}=5{\cdot}8 \text{ feet nearly.}$$

By formula (5)

$$v=\sqrt{\frac{2380\times170\times{\cdot}5}{5780+54\times{\cdot}5}-\frac{1}{12}\cdot\frac{5780}{5780+54\times{\cdot}5}}=5{\cdot}81.$$

By formula (7)

$$v=47{\cdot}9\sqrt{\frac{170\times{\cdot}5}{5780}}=5{\cdot}8.$$

By formula (8)

$$v=\sqrt{\frac{2380\times170\times{\cdot}5}{5780}-\frac{1}{12}}=5{\cdot}82.$$

It will be observed how very nearly these results correspond with one another.

When d is expressed in inches, all the other dimensions being in feet, formula (2) becomes

$$v=\sqrt{\frac{191{\cdot}2dh}{l+4{\cdot}5d}}\ldots(9).$$

To find the quantity of water discharged.

47. Let Q=the number of cubic feet of water discharged per second, d=the diameter of the pipe in inches; then

$$Q=\frac{1}{4}\pi\left(\frac{d}{12}\right)^2v$$

$$=\frac{1}{576}\pi d^2v\ldots(10).$$

If q=the discharge in gallons for 24 hours, then, assuming that there are 6·2322 gallons in 1 cubic foot, we have

$$q=\frac{1}{576}\pi d^2 v\times 86400\times 6\cdot 2322$$

$$=2936\cdot 86 d^2 v \ldots (11).$$

Or, for convenience of calculation, we have in logarithms,

$$\log q=3\cdot 4678835+2\log d+\log v \ldots (12).$$

Example. How many gallons of water would the pipe of Example 1. Art. **46.**, deliver in 24 hours?

Here, taking $v=5\cdot 8$, $d=6$ in., we have by eq. (10)

$$q=2936\cdot 86\times 6^2\times 5\cdot 8=613216 \text{ gals. in 24 hours.}$$

Or by eq. (12)

$$\log q=3\cdot 4678835+2\log 6+\log 5\cdot 8=5\cdot 7876140.$$

$$\therefore\ q=613210 \text{ gallons in 24 hours.}$$

To find the diameter of pipes.

Substituting in eq. (10) the value of v given in eq. (9), we get

$$Q=\frac{1}{576}\pi d^2\sqrt{\frac{191\cdot 2 dh}{l+4\cdot 5 d}},$$

$$\therefore\ d=\left\{\frac{175\cdot 81 Q^2(l+4\cdot 5 d)}{h}\right\}^{\frac{1}{5}} \ldots (13);$$

Or by logarithms,

$$\log d=\tfrac{1}{5}\{2\cdot 2450532+2\log Q+\log(l+4\cdot 5 d)-\log h\} \ldots (14),$$

where d is in units of inches, all the other terms being in feet.

When the pipes are very long, or when d is small as compared with l, eq. (14) becomes

$$\log d=\tfrac{1}{5}\{2\cdot 2450532+2\log Q+\log l-\log h\} \ldots (15).$$

The value of d can only be obtained from eq. (13) by successive approximation. Thus, when considerable accuracy is required, the value of d must be first calculated by eq. (15), and then substituted in eq. (14), which will give a first approximate value of d, and this again substituted in eq. (14) will give the value of d more nearly; and so on to any degree of accuracy; but in general the first approximate value will be found sufficiently accurate for all practical purposes.

Example. What must be the diameter of a pipe which shall deliver 25,000 gallons of water per hour, when the length of the pipe is 2500 feet, and the head of water 225 feet?

Here $h=225$, $l=2500$, and to find Q, we have

$$\text{No. gals. delivered per sec.} = \frac{25000}{60\times 60},$$

$$\therefore \text{ No. c. ft. delivered per sec., or } Q = \frac{2500}{3600\times 6\cdot 232}$$

$$=1\cdot 1145 \text{ nearly.}$$

Now from eq. (15), we have

$$\log d = \tfrac{1}{5}\{2\cdot 2450532+2\log 1\cdot 1145+\log 2500-\log 225\}$$

$$=\tfrac{1}{5}\{\bar{1}\cdot 9850309+3\cdot 3979400\}=\cdot 67659.$$

$$\therefore d=4\cdot 7489 \text{ inches.}$$

Now substituting this value of d in eq. (14), we get

$$\log d=\tfrac{1}{5}\{\bar{1}\cdot 9850309+\log(2500+4\cdot 5\times 4\cdot 7489)\}$$

$$=\cdot 6773311\,;\ \therefore d=4\cdot 757 \text{ inches.}$$

Here it is not necessary to carry the approximation any further.

48. The formula given by Weisbach for the flow of water in long pipes is

$$h=\left(1\cdot 505+\alpha\frac{l}{d}\right)\frac{v^2}{2g}\ \ldots\ (1);$$

from this equation, we get

$$v=\frac{\sqrt{2gh}}{\sqrt{1\cdot 505+\alpha\frac{l}{d}}}\ \ldots\ (2),$$

where α is a quantity depending to a certain extent upon v, viz.,

$$\alpha=\cdot 01482+\frac{\cdot 017963}{\sqrt{v}}\ \ldots\ (3).$$

Here the valve of v must be determined by successive approximations.

Example. Taking the data of Example 1, Art. **46.**, we first assume $v=5\cdot 3$; then from eq. (3) we find $\alpha=\cdot 01482+\dfrac{\cdot 017963}{\sqrt{5\cdot 3}}$ $=\cdot 02263$; and substituting this in eq. (1), we have

$$v=\frac{\sqrt{64\frac{1}{3}\times 170}}{\sqrt{1\cdot 505+\cdot 02263\times\frac{5780}{\cdot 5}}}=6\cdot 4.$$

Substituting 6·4 for v and going over the whole work again, we find $v=6{\cdot}5$, which gives the value true to the first decimal place. This result is about $\frac{1}{10}$ greater than that derived from eq. (3), Art. **46.**

For high velocities this formula gives the value of v in excess of the values derived from eqs. (3) and (5), Art. **46.**; and the converse is the case for low velocities. Thus if $v=2{\cdot}25$, $d=\frac{1}{2}$, $l=1000$, eq. (1) gives $h=4{\cdot}34$; and these values substituted in eq. (5) give $v=2{\cdot}17$, where there is a difference of about $\frac{1}{30}$. If $l=100$, the other data being the same, eq. (1) gives $h={\cdot}542$, which substituted in eq. (5) gives $v=2{\cdot}2$ nearly, where there is a difference of about $\frac{1}{40}$. If $v=3$, $d=\frac{1}{4}$, $l=100$, eq. (1) gives $h=1{\cdot}62$; and these values substituted in eq. (5) give $v=2{\cdot}84$, and in eq. (3) give $v=2{\cdot}86$, where the first value of v is in excess of the last two values by about $\frac{1}{20}$. If $v=1$, $d=\frac{3}{4}$, $l=100$, eq. (1) gives $h={\cdot}0918$; and from eq. (5) we find $v=1{\cdot}02$, and from eq. (3) $v=1{\cdot}05$, where the two last values are in excess of the first value.

Resistances of Water in passing through Contractions.

49. Owing to the sudden change in the velocity of the current of water, a loss of vis viva, or accumulated work, always takes place at all abrupt alterations in the dimensions of the pipe. Thus let BC represent an abrupt alteration in the dimensions of the pipe ABCD, then as the fluid in the smaller pipe has a greater velocity than the fluid in the larger one, a sudden change of velocity will take place in the passage of the fluid from the one pipe to the other, and this change of velocity will occasion a loss of accumulated work, in the same manner as when two inelastic bodies impinge upon each other.

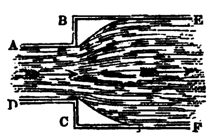


Fig. 21.

Let v_1=the velocity of the water in the pipe ABCD; v_2=the velocity of the water in BEFC; W_1=the weight of water discharged from ABCD per second; and W_2 =the weight of the whole mass of water in BCFE.

Now after impact the mass of fluid W_1 moves on with the whole mass of W_2, in the pipe BCFE, with the common velocity v_2; and moreover, it will be observed, that W_2 is very great as compared with W_1; hence we have from eq. (7), Art. **202.**, page 187.

Work lost by the water in passing from the one pipe to the other $\doteq \frac{(v_1-v_2)^2}{2g} W_1$.

Now if h_1=the head of water corresponding to this loss of work, we have

$$\text{Work lost}=h_1 w_1,$$

$$\therefore\ h_1 w_1=\frac{(v_1-v_2)^2}{2g}w_1,$$

$$\therefore\ h_1=\frac{(v_1-v_2)^2}{2g}\ \ldots\ (1),$$

that is to say, THE LOSS OF HEAD OF WATER, ARISING FROM A SUDDEN CHANGE OF VELOCITY, IS MEASURED BY A COLUMN OF FLUID CORRESPONDING TO THE LOSS OR CHANGE OF VELOCITY.

Let a_1=the area of the section of the pipe ABCD; a_2=the area of the section of the pipe BCFE; and β_1=the coefficient of resistance; then

$$\frac{v_1}{v_2}=\frac{a_2}{a_1},\ \text{and}\ \therefore\ v_1=\frac{a_2}{a_1}v_2,$$

substituting in eq. (1), we get

$$h_1=\left(\frac{a_2}{a_1}-1\right)^2\cdot\frac{v_2^2}{2g}\ \ldots\ (2),$$

$$\therefore\ \beta_1=\left(\frac{a_2}{a_1}-1\right)^2\ \ldots\ (3).$$

This theoretical deduction appears to agree very nearly with the results of experiment.

50. When the passage from one pipe to the other is rounded off, as in the annexed cut, and the difference in the sections is not considerable, the loss of work, as shown by experiment, is very small.

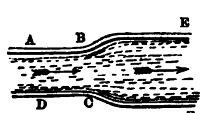


Fig. 22.

51. In like manner, when a contraction takes place in a pipe, the loss of work is very much diminished by rounding off the parts, as shown in the annexed cut.


Fig. 23.

Example. The diameters of the pipes, as shown in *fig.* 21., are 5 and 10 inches respectively, and the velocity of the water in the larger pipe is 4 feet; required the loss of head of water, and also the coefficient of resistance.

Here $\frac{a_2}{a_1}=\frac{10^2}{5^2}=4$, and $v_2=4$, hence we have from eq. (2)

$$h_1=(4-1)^2\cdot\frac{4^2}{2\times 32\frac{1}{6}}=2\frac{1}{4}\ \text{feet},$$

which is the loss of head of water due to the abrupt alteration in the size of the pipe. From eq. (3), we have

$$\beta_1=(4-1)^2=9,$$

which is the coefficient of resistance.

Resistance of Water in passing through bent Pipes.

52. The loss of work, in this case, is due to the change which takes place in the *direction* of the motion of the water; and this loss of work, or what amounts to the same thing, the loss of head of water, in curved pipes, increases with the square of the velocity and also with the angle of deviation. Let AB and AC be the axes of the pipes; O the centre of the arc EF forming the curve of the bend; CAD the angle of diviation, which is obviously equal to the angle EOF. Put $\theta=\angle \text{CAD}=\angle \text{EOF}$; v=the velocity of the water; then

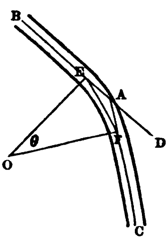


Fig. 24.

$$\text{the loss of head, } h=e\cdot\frac{\theta}{180}\cdot\frac{v^2}{2g}\ldots(1),$$

where e is not strictly a constant, for it is evident that, other things being the same, the loss of head will be diminished by increasing the radius of curvature, OE or OF, of the curve forming the bend of the pipe. If R=OE and r=the radius of the pipe; then, according to Weisbach,

$$e=\cdot131+1\cdot847\left(\frac{r}{\text{R}}\right)^{\frac{7}{2}}\ldots(2),$$

Thus for bends whose radius is 4 times the radius of the pipe

$$e=\cdot131+1\cdot847\times(\tfrac{1}{4})^{\frac{7}{2}}=\cdot1455.$$

If $\frac{r}{\text{R}}=\frac{1}{10}$, then the latter part of this expression, viz., $1\cdot847\left(\frac{r}{\text{R}}\right)^{\frac{7}{2}}$ is less than $\cdot0006$; therefore for all cases where R exceeds 10 times the radius of the pipe, $e=\cdot131$; for such case, therefore, we have

$$\text{loss of head, } h=\cdot131\,.\,\frac{\theta}{180}\cdot\frac{v^2}{2g}$$

$$=\cdot00073\,\theta\frac{v^2}{2g}\ldots(3).$$

Example. The diameter of a curved pipe is 6 inches, the angle of deviation 40°, the velocity of the water 8 feet per second, and the radius of the curve forming the bend 30 in.; required the loss of head of water due to the resistance of the bend of the pipe.

Here $\frac{r}{R}=\frac{3}{30}=\frac{1}{10}$, hence eq. (3) applies to the present example;

$$\therefore \text{ loss of head} = \cdot 00073 \times 40 \times \frac{8^2}{64\frac{1}{3}} = \cdot 0292 \text{ ft. nearly.}$$

53. When the bend of the pipe has the form of a knee, as shown in *fig.* 25., the loss of head is much greater than when the bend is curved: thus when the bend is a rectangular knee, the loss of head according to Weisbach is $\cdot 956\,\frac{v^2}{2g}$, which is nearly equal to the height due to the velocity.

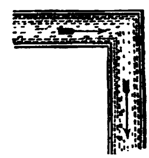


Fig. 25.

General Formula taking all the Resistances to the Flow of the Water into Account.

54. In addition to the resistances expressed in eq. (3), Art. **46.**, let β_1=the coefficient of resistance for contractions (see eq. (3), Art. **49.**); β_2=the coefficient of resistance for bendings (see eq. (3), Art. **52.**, &c.); then we find after the method of deriving eq. (3), Art. **46.**,

$$h=(e_1 v+e_2 v^2)\frac{l}{d}+\frac{v^2}{2g}+\beta\frac{v^2}{2g}+\beta_1\frac{v^2}{2g}+\beta_2\frac{v^2}{2g}$$

$$=(e_1 v+e_2 v^2)\,\frac{l}{d}+(1+\beta+\beta_1+\beta_2)\frac{v^2}{2g}\ldots(1);$$

where the constants, as assigned in Art. **46.**, are as follows: $e_1=\cdot 00007$; $e_2=\cdot 00042$; $\beta=\cdot 5$; β_1 as given eq. (3), Art. **49.**; $\beta_2=\cdot 00073\,\theta$ under the limitations explained in Art. **52.**

Neglecting e_1 and putting $e=2g\times e_2=\cdot 028$ (see eq. (1), Art. **45.**), we get

$$h=\left(e\frac{l}{d}+1+\beta+\beta_1+\beta_2\right)\frac{v^2}{2g}\ldots(2).$$

From these equations v, Q, &c., may be determined.

Example. In the pipe of Example 1., Art. **46.**, there are forty bends having each an angle of 50° deviation, with a radius of

curvature exceeding ten times the radius of the pipe; required the velocity of efflux.

Here by eq. (2) we have

$$e = \cdot 028\,;\ \beta = \cdot 5\,;\ \beta_2 \times 40 = \cdot 00073 \times 50 \times 40 = \cdot 0365 \times 40 = 1\cdot 46\,;$$

$$l = 5780\,;\ d = \cdot 5\,;\ h = 170\,;$$

$$\therefore\ 170 = \left(\cdot 028 \times \frac{5780}{\cdot 5} + 1 + \cdot 5 + 1\cdot 46\right)\frac{v^2}{64\frac{1}{3}}\,;$$

$$\therefore\ 170 = 326\cdot 64 \times \frac{v^2}{64\frac{1}{3}}\,;$$

$$\therefore\ v = 5\cdot 7885 \text{ feet per second.}$$

55. It will be seen from these calculations, that in long tubes the principal resistance to the flow of the water is that of friction; it is therefore especially desirable that the constants of friction should be determined with the greatest precision.

It appears to the Author that the general formula for friction, assumed in Art. **46.**, is not sufficiently exact, and that the experimental data would be more completely represented by the relation

$$h = (e_1 v^{\alpha} + e_2 v^2)\frac{l}{d},$$

where the constants e_1, e_2, and α have to be assigned by experiment.

FLOW OF WATER IN RIVERS, CANALS, AND OPEN CHANNELS.

Mean Velocity, &c.

56. Let ABCD represent a longitudinal section of a stream or any channel not filled with water; DE a horizontal line perpendicular to the verticals AD and BE; DC$=l$, the length in feet; CE$=h$, the fall of the stream in feet for the length l; $\angle$CDE$=\theta$, the angle of the fall; $a=$the transverse section at BC in sq. ft., and V=its mean velocity; $p=$the wetted perimeter of the transverse section; $r=\frac{a}{p}$, the mean depth, or, as it may be called,

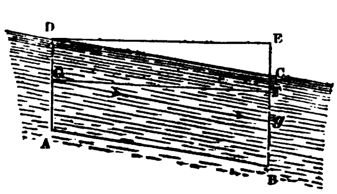


Fig. 26.

the mean radius of the section; Q=the no. c. ft. of water flowing through the section at BC per second; then

$$\sin\theta=\frac{h}{l}, \text{ the fall for each foot,}$$

$$\text{and } Q=aV.$$

If V_1=the mean velocity at any part AD having the section a_1; then when the stream has attained the condition of permanent flow, we also have

$$Q=a_1V_1;$$

for in this case the quantity of water flowing through these sections is the same;

$$\therefore\ aV=a_1V_1;\ \therefore\ \frac{V}{V_1}=\frac{a_1}{a}.$$

57. Owing to the friction of the water against its banks and the bottom of its channels, the greatest velocity of the stream is at the middle of its surface; from this point the velocity decreases towards the sides and bottom, where it is least. If v=the greatest velocity, or the velocity at the surface, then, according to Prony,

$$V=\frac{v(v+7{\cdot}77)}{v+10{\cdot}33}.$$

Example. If $v=4$ ft.; then

$$V=\frac{4(4+7{\cdot}77)}{4+10{\cdot}33}=3{\cdot}3 \text{ ft. nearly.}$$

WHEN THE VELOCITY OF THE STREAM IS UNIFORM.

58. When water flows in an open channel the velocity goes on increasing so long as the accelerating force exceeds the resisting force of friction; but when these forces are equal to each other, by the principle explained in Art. **286**., p. 237., the velocity of the stream becomes uniform. In this case, therefore, the head of water due to the fall must be equal to the column of fluid due to the resistance of friction; hence we have, as in Art. **46**.

$$h=(e_1V+e_2V^2)\frac{l}{r},$$

where e_1, e_2 are the coefficients of velocity due to the resistance of friction. By an obvious reduction this expression becomes

$$h=\alpha(vV^2+\beta V)\frac{l}{r}.$$

Or substituting $\frac{a}{p}$ for r, we have

$$h=\alpha(v^2+\beta v)\frac{pl}{a}\ldots(1).$$

Taking the constants in accordance with Eytelwein's reduction of the ninety-one observations and experiments made by Du Buat and others, we have

$$\alpha=\frac{\cdot0035855}{g}=\cdot0001114, \text{ and } \beta=\cdot2028.$$

59. From this equation v may be found by the solution of a quadratic; but the following method of approximation gives the value of v more simple and at the same time exact enough for practical use.

By reduction, we get

$$v\left(1+\frac{\beta}{v}\right)^{\frac{1}{2}}=\sqrt{\frac{ah}{\alpha pl}};$$

expanding by the binomial theorem, we have

$$\left(1+\frac{\beta}{v}\right)^{\frac{1}{2}}=1+\frac{\beta}{2v}-\frac{1}{8}\left(\frac{\beta}{v}\right)^2+\&c.$$

Now when v is not less than $\frac{1}{10}$ we may neglect all the terms in this expansion after the second; hence by substitution, we get

$$v\left(1+\frac{\beta}{2v}\right)=\sqrt{\frac{ah}{\alpha pl}};$$

$$\therefore\ v=\sqrt{\frac{ah}{\alpha pl}}-\frac{\beta}{2}\ldots(2).$$

Substituting the values of the constants α and β as above given, we get

$$v=94\cdot7\sqrt{\frac{ah}{pl}}-\cdot1014\ldots(3).$$

When v is considerable the last constant in this equation may be neglected, and in this case, we have

$$v=94\cdot7\sqrt{\frac{ah}{pl}}\ldots(4)$$

For the number of cubic feet of water flowing through the channel per second, we have

$$Q = a\mathrm{v} = a\left\{\sqrt{\frac{ah}{\alpha pl}} - \frac{\beta}{2}\right\} \ldots (5)$$

$$= a\left\{94{\cdot}7\sqrt{\frac{ah}{pl}} - {\cdot}1014\right\}.$$

Eq. (1) substantially agrees with the formula given by Eytelwein.*

Reducing the constants taken by Prony to English measure, we find $\alpha = {\cdot}0000911$, and $\beta = {\cdot}48$.

Now substituting these in eq. (2) and reducing, we get

$$\mathrm{v} = 104\sqrt{\frac{ah}{pl}} - {\cdot}24 \ldots (6),$$

which for ordinary velocities does not differ much from eq. (3).

It appears that Prony's values of the constants were taken in relation to comparatively small streams.

Example 1. Let $a = {\cdot}5$, $p = 2$, $h = 5$, $l = 1100$; required v and Q.

Here by eq. (3), we have

$$\mathrm{v} = 94{\cdot}7\sqrt{\frac{{\cdot}5 \times 5}{2 \times 1100}} - {\cdot}1014 = 3{\cdot}08 \text{ feet},$$

which is the mean velocity of the stream.

And by eq. (5)

$$Q = {\cdot}5 \times 3{\cdot}08 = 1{\cdot}504 \text{ cubic feet},$$

which is the discharge of water per second.

Again by eq. (6) we have

$$\mathrm{v} = 104\sqrt{\frac{{\cdot}5 \times 5}{2 \times 1100}} - {\cdot}24 = 3{\cdot}26 \text{ feet},$$

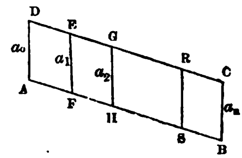
which does not differ much from the result before found.

When the Velocity of the Stream is accelerated.

60. In this case the work due to the fall of the stream for a given distance will be equal to the work destroyed by friction together with the work accumulated in the water for that distance.

* Experimentalists have assigned different values to the constants in these equations; but this circumstance does not affect the general formulæ given in eqs. (2) and (5).

Let G, g be the centres of gravity of the sections (see *fig*. 26., p. 301.); draw the horizontal line GF; then Fg will be the fall of the water for the distance DC; but when the depth of the stream does not relatively vary much, then Fg will be nearly equal CE, that is, the fall of the centre of gravity of the stream will be nearly equal to the fall of its surface. Put v_n, v_0=the mean velocities of the stream at the sections BC and AD respectively; a_n, a_0=the areas of these sections; Q=the number of cubic feet of water flowing through the channel per second; w=the weight of cubic foot of water. Let the length DC of the stream be divided into n equal parts, DE=EG= ... =RC, and let a_1, a_2, . . be the areas of these sections as shown in the annexed cut, v_1, v_2, . . . the mean velocities of the stream at these sections, and p_1, p_2, . . ., p_n, the mean wetted perimeters; h_1, h_2, . . ., h_n=the falls of water in the portions DF, EH, . . ., RB respectively; h=the total fall from AD to BC.



Now for the portion DF. If the stream flowed through DF with the velocity v_0, the mean perimeter of the channel being p_1 and the mean section $\frac{1}{2}(a_0+a_1)$, we should have by Art. **45.**,

$$\text{Work due to friction} = \alpha v_0^2 \frac{p_1}{\frac{1}{2}(a_0+a_1)} \frac{l}{n} \times Qw;$$

and if it flowed with the velocity v_1, we should have,

$$\text{Work due to friction} = \alpha v_1^2 \frac{p_1}{\frac{1}{2}(a_0+a_1)} \frac{l}{n} \times Qw;$$

but the former expression is less than the true amount of work, while the latter is greater than it is, hence the mean of these results will give the work approximately,

$$\therefore \text{ Work due to friction} = \frac{\alpha l}{n} \cdot \frac{p_1}{(a_0+a_1)}(v_0^2+v_1^2) \times Qw,$$

but $v_0 = \frac{Q}{a_0}$, and $v_1 = \frac{Q}{a_1}$,

$$\therefore \text{ Work due to friction} = \frac{\alpha l Q^2}{n} \cdot \frac{p_1}{a_0+a_1}\left(\frac{1}{a_0^2}+\frac{1}{a_1^2}\right) \times Qw.$$

By eq. (1), Art. **49.**, we have

$$\text{Work accumulated at EF} = \frac{v_1^2 - v_0^2}{2g} \times Qw,$$

$$= \frac{Q^2}{2g}\left(\frac{1}{a_1^2} - \frac{1}{a_0^2}\right) \times Qw.$$

$$\text{Work due to the fall of water} = h_1 \times Qw.$$

Now we have

Work due to the fall = Work accd. in the water + Work due to friction;

$$\therefore\ h_1 \times Qw = \frac{Q^2}{2g}\left(\frac{1}{a_1^2}-\frac{1}{a_0^2}\right)\times Qw + \frac{\alpha l Q^2}{n}\cdot\frac{p_1}{a_0+a_1}\left(\frac{1}{a_0^2}+\frac{1}{a_1^2}\right)\times Qw,$$

$$\therefore\ h_1 = \frac{Q^2}{2g}\left(\frac{1}{a_1^2}-\frac{1}{a_0^2}\right)+\frac{\alpha l Q^2}{n}\cdot\frac{p_1}{a_0+a_1}\left(\frac{1}{a_0^2}+\frac{1}{a_1^2}\right)$$

In like manner, we find

$$h_2 = \frac{Q^2}{2g}\left(\frac{1}{a_2^2}-\frac{1}{a_1^2}\right)+\frac{\alpha l Q^2}{n}\cdot\frac{p_2}{a_1+a_2}\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}\right),$$

&c. = &c.

Now $h=h_1+\ \ldots\ +h_n$; hence we get by addition,

$$h = Q^2\left[\frac{1}{2g}\left(\frac{1}{a_n^2}-\frac{1}{a_0^2}\right)+\frac{\alpha l}{n}\left\{\frac{p_1}{a_0+a_1}\left(\frac{1}{a_0^2}+\frac{1}{a_1^2}\right)+\ldots\right.\right.$$
$$\left.\left.+\frac{p_n}{a_{n-1}+a_n}\left(\frac{1}{a_{n-1}^2}+\frac{1}{a_n^2}\right)\right\}\right]\ldots(4).$$

If $n=1$, then we simply have

$$h = Q^2\left\{\frac{1}{2g}\left(\frac{1}{a_1^2}-\frac{1}{a_0^2}\right)+\frac{\alpha l p_1}{a_0+a_1}\left(\frac{1}{a_0^2}+\frac{1}{a_1^2}\right)\right\}\ldots(5).$$

From this equality Q is readily determined.

In a prismoidal channel, we may assume, as a sufficient approximation to the truth, the surface line of the water to be straight; and then, from this assumption, the sections a_1, a_2, &c., with their perimeters p_1, p_2, &c., may be calculated by geometry when the extreme sections a_n, a_0 are given. To determine h from eq. (4), therefore, we must have given V, Q, V_0, a_0, l, and the form of the transverse section of the channel: for from V and Q we may find a_n and consequently p_n; from a_0 we may find p_0; and then from a_n, p_n, a_0, p_0, and l, we may find the equidistant sections a_1, a_2, &c. with their perimeters p_1, p_2, &c.

Example 1. A stream has a fall of ·81 feet in 300 feet, the upper transverse section contains 70 sq. feet, that of the lower section 60 sq. feet, and the mean perimeter 42 feet; required the quantity of water, Q, flowing through the channel per second.

Here $n=1$, $h=·81$, $l=300$, $a_0=70$, $a_1=60$, $p_1=42$, $\alpha=·0001114$, see Art. **58**.; hence we have from eq. (4),

$$\cdot 81 = Q^2 \left[\frac{1}{64\frac{1}{3}} \left(\frac{1}{60^2} - \frac{1}{70^2} \right) + \cdot 0001114 \right.$$

$$\left. \times 300 \left\{ \frac{42}{70+60} \left(\frac{1}{70^2} + \frac{1}{60^2} \right) \right\} \right],$$

$$\therefore \; Q = \sqrt{\frac{\cdot 81}{\cdot 00000634}} = 360 \text{ cubic feet.}$$

And the mean velocity at the lower section $= \frac{Q}{a} = \frac{360}{60} = 6$ feet.

THE BEST FORM OF THE TRANSVERSE SECTION OF A STREAM.

61. The best form of the section must be that which presents the least resistance to a given quantity of water flowing through the channel. Now it has been shown, Art. **45.**, that the resistance of friction varies directly as the perimeter and inversely as the area of the section, and when the area of the section is constant it will vary directly as the perimeter; consequently the best form of the section will be that which has the least perimeter for a given area. Hence, the circle, and regular figures with a considerable of sides, are best adapted for the form of the transverse section of closed currents; but in open channels the upper water line must not be included in the perimeter. Of all rectangular forms of sections the half square ABCD is the best for open channels; of all circles the semicircle ACB is best; and of all trapezoidal sections the semi-hexagon ABCD is the best; and so on

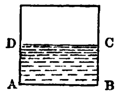


Fig. 27.

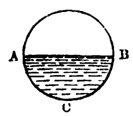


Fig. 28.

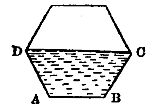


Fig. 29.

to the other cases. But for equal flows of water the semi-circle will have less friction than the semi-hexagon, and this latter less than the semi-square.

Now in canals which are not walled, the slope of the sides depends on the nature of the materials through which the water flows; hence the following problem becomes important:

To determine the best form of a trapezoidal section of a canal when the slope of the sides is given.

62. Let ABCD be the section; put
x=AB the breadth of the bottom;
y=BE the vertical depth;
$\theta=\angle$CBE, the given angle which the side makes with the vertical;
a=the given area of the section ABCD.

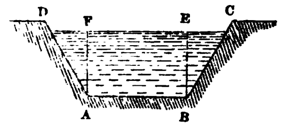


Fig. 30.

$$\text{Perimeter} = AB + 2BC$$
$$= x + 2 \sec \theta y;$$

but we have for the area of the section,

$$xy + \tan \theta y^2 = a;$$

$$\therefore \; x = \frac{a - \tan \theta y^2}{y} \ldots (1);$$

therefore by substitution, we have

$$\text{Perimeter} = \frac{a - \tan \theta y^2}{y} + 2 \sec \theta y = \text{a minimum.}$$

By differentiation, &c., we get

$$y^2 = \frac{a}{2 \sec \theta - \tan \theta} = \frac{a \cos \theta}{2 - \sin \theta} \ldots (2),$$

whence the perpendicular depth BE is determined, and this value substituted in eq. (1) will give the breadth of the bottom AB.

Example. Required the dimensions of the transverse section of a canal whose banks have a slope of 45° with the vertical, and which is to conduct 108 cubic feet of water per second, with the mean velocity of 3 feet.

Here $\theta = 45°$, and $a = \frac{Q}{V} = \frac{108}{3} = 36$ sq. ft.

$$\therefore \; y^2 = \frac{36 \cos 45°}{2 - \sin 45°} = 19{\cdot}7, \text{ and } y = 4{\cdot}4 \text{ ft.} = BE.$$

Substituting these values in eq. (1), we find

$$x = 3{\cdot}7 = AB \text{ the breadth of the bottom.}$$

Greatest Velocity of Water in Canals.

63. The velocity of a stream should not be so slow as to allow the channel to be choked up with weeds and depositions of slime

or sand. A velocity of $\frac{1}{2}$ to $\frac{3}{4}$ of a foot is requisite for preventing the deposition of slime, and from 1 to $1\frac{1}{4}$ feet to prevent the deposition of sand. On the other hand the velocity of a stream should not be so rapid as to wash away the material composing the channel. The maximum velocity of the stream depends upon the nature of the material forming the bed of the channel. On a slimy bed the mean velocity should not exceed $\frac{1}{4}$ foot; on clay $\frac{1}{2}$ a foot; on sand 1 foot; on gravel 2 feet; on conglomerate 5 feet; and on stone 10 feet.

General Remarks relative to Water Pipes.

64. An enlargement should be made in the pipe at the part of ingress as well as egress; and when any change takes place in the diameter of the pipe, the parts at the junction should be rounded off. See Arts. **31.** and **50.**

65. It may be advantageous to increase the size of the pipe at any considerable bendings.

66. When different streams of water meet, they should if possible have the same velocity.

67. At junctions the currents should be made to flow on together as nearly as possible in the same direction, and therefore at such junctions the smaller pipe should be curved as in the accompanying figure.

Fig. 31.

68. When a pipe receives a considerable accession of water from any branch pipe, the main pipe after this accession may be enlarged with advantage so as to maintain the volocity of the fluid unchanged; at the same time, excepting in extreme cases, there would not be much additional loss occasioned by having the main pipe of the same dimensions throughout its length.

69. At all considerable rises a provision should be made for clearing the pipe of air. Thus at the bending B, the air, which is time after time disengaged from the water, will accumulate, and unless some provision is made for its escape it will obstruct the flow of the current.

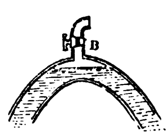


Fig. 32.

70. Piezometers or pressure gauges are very useful for ascertaining the place in a pipe where an obstruction may have occurred. They also afford data

for determining the coefficient of resistance to the motion of the water.

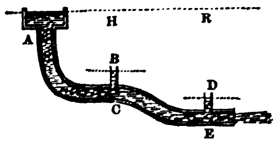


Fig. 33.

Let piëzometers be placed at C and E; and let CB$=z$ the height of the column of water in the tube; CH$=h$ the fall at C; ED$=z_1$ the height of the column at E; ER$=h_1$ the fall at E; $l=$AC the length of the pipe at C; $l_1=$CE the length of the pipe between C and E; then we have

The loss of head at C$=$BH$=h-z$,

but the loss of head, due to the resistance of ingress and friction, is given in eq. (1) Art. **45.**,

$$\therefore\ h-z=\left(1+\beta+e\frac{l}{d}\right)\frac{v^2}{2g}\ \ldots (1),$$

whence z is readily determined.

In like manner we have

$$h_1-z_1=\left(1+\beta+e\frac{l+l_1}{d}\right)\frac{v^2}{2g},$$

hence we have by subtraction,

$$e\frac{l_1}{d}\,\frac{v^2}{2g}=h_1-h+z-z_1\ \ldots (2);$$

from this equation the coefficient of resistance, e, is readily determined.

CHAP. XXI.

ON HYDRAULIC ENGINES.

WORK OF WATER WHEELS.

71. WHEN a stream of water reaches the paddles of a wheel which has a certain velocity, the work imparted to the wheel by the water is expressed by eq. (5) Art. **41.**; but if the water descends with the paddle there is an additional portion of work done on the wheel due to the mean vertical space h through which the water falls; hence we have

$$\text{Work applied to the wheel per sec.} = \frac{W}{2g}\{v^2-(v-v_1)^2\} + Wh.$$

Now this work applied must be equal to the useful work, U, which the wheel does, added to the accumulated work remaining in the water after leaving the paddles. If v_2=the velocity of the water after it has left the paddles, then the work remaining in the water will be equal to $\frac{W \times v_2^2}{2g}$; hence we have,

$$U + \frac{W \times v_2^2}{2g} = \frac{W}{2g}\{v^2-(v-v_1)^2\} + Wh,$$

$$\therefore \quad U = Wh + \frac{W}{2g}\{2vv_1 - v_1^2 - v_2^2\} \; . \; . \; . \; (1),$$

which is the general expression for the work done by a water-wheel.

We here suppose that the water impinges upon the paddles perpendicularly.

Work of Undershot Wheels.

72. In this case the water has no perpendicular fall, and the velocity of the water upon leaving the paddle is the same as the velocity of the paddle itself; hence we have $h=0$, $v_2=v_1$, and then eq. (1), Art. **71.**, becomes

$$U = \frac{W}{g}(v-v_1)v_1 \; . \; . \; . \; (2);$$

or, we have, by introducing m a constant coefficient to be determined by experiment,

$$U = \frac{mW}{g}(v-v_1)v_1 \; . \; . \; . \; (3),$$

where m=·6, according to Morin's experiments.

73. To find the relation of v and v_1 so that the useful work of the wheel may be a maximum.

Now eq. (3), may be written as follows:

$$U = \frac{mW}{g}\{\tfrac{1}{4}v^2 - (\tfrac{1}{2}v - v_1)^2\}$$

where U is evidently a maximum when

$$\tfrac{1}{2}v - v_1 = 0; \text{ or when } v_1 = \tfrac{1}{2}v,$$

that is to say, when *the velocity of the wheel is one half the velocity of the water.*

WORK OF BREAST WHEELS.

74. In this case the height h of the fall is the vertical height of the point at which the water meets the paddles above the point where it leaves them; and as in the foregoing case $v_2 = v_1$; hence eq. (1), Art. **71**, becomes

$$U = Wh + \frac{W}{g}(v - v_1)v_1 \ldots (4);$$

or introducing the constant m,

$$U = mW\left\{h + \frac{1}{g}(v - v_1)v_1\right\} \ldots (5),$$

where $m = \cdot 74$, according to Morin.

Theoretically the maximum work takes place, as in the foregoing case, when $v_1 = \frac{1}{2}v$. But Morin found, by experiments, that the efficiency of the wheel is not much affected by changes in its velocity. This is owing to the circumstance that the useful work is principally dependent upon the term Wh, see eq. (4), and not upon the other term in the formula which alone is affected by the velocity of the wheel. Hence the great advantage of this wheel is, that it may be worked, without materially impairing its efficiency, with velocities varying from $\frac{1}{3}v$ to $\frac{2}{3}v$.

WORK OF OVERSHOT WHEELS.

75. In this wheel the mean vertical height h, which the water falls, is nearly equal to the diameter of the wheel, and as in the case of the breast wheel, $v_2 = v_1$; hence eqs. (4) and (5), Art. **74**. also apply to overshot wheels. But in this case, $m = \cdot 78$.

WORK OF PONCELET'S UNDERSHOT WHEEL.

76. In the common undershot water wheel, the paddles are flat, whereas in Poncelet's wheel they have a curved shape, AB; so that the direction of the curve at A, where the water first meets the paddle, is the same as the direction of the stream. By this ingenious contrivance, the water rolls up the curved incline AB, without meeting with any sudden obstruction calculated to occasion a loss of work. The channel has a depression at the point where the water falls from the paddles.

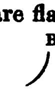


Fig. 34.

Let v be the velocity of the stream, and v_1 that of the wheel, then since the point A of the paddle is moving away from the stream, the water will flow upon the paddle with the velocity $v-v_1$, and will continue to run up the curved incline until it has lost its motion, it will then descend, acquiring in its descent the same velocity as that which it had in its ascent, but in a contrary direction. If the wheel were not moving $v-v_1$ would be exactly the velocity of discharge from right to left, but the paddle is moving with the water from right to left with the velocity v_1, therefore the absolute velocity of the water upon leaving the paddle will be $v-v_1-v_1=v-2v_1$.

Now we have in this wheel the following relation:

Accd. work in the water=Work done on the wheel+Accd. work in the water after leaving the paddles.

$$\therefore \frac{w \times v^2}{2g} = U + \frac{w(v-2v_1)^2}{2g},$$

$$\therefore U = \frac{2w}{g}(v-v_1)v_1 \ldots (6).$$

Comparing this expression with eq. (2), Art. **72.**, we find that the work performed by Poncelet's wheel is double that of the common undershot wheel.

By Art. **73.** we find that there will be the greatest work done when $v_1=\frac{1}{2}v$. This conclusion may also be established by the following mode of reasoning: —

All the work will have been taken out of the water, when its motion upon leaving the paddle is nothing, that is, when $v-2v_1=0$ or $v=2v_1$. In this case, the water having lost all its motion, will simply drop from the paddle, and the work done upon the wheel will be equal to the work accumulated in the water of the stream. Moreover, it appears that this maximum condition is fulfilled when the velocity of the stream is double that of the wheel. However the distinguished inventor states, that, in practice, the velocity of the water, in order to produce its maximum effect, ought to be about $2\frac{1}{2}$ times that of the wheel, and that then the modulus of the wheel is about ·7.

BARKER'S MILL, OR THE REACTION WHEEL. CENTRIFUGAL PUMP. PUMPS.

77. In the best form of the reaction wheel, the vanes or arms are curved and the water is projected from them in lines forming

tangents to the circle described by the orifice. The velocity of the issuing fluid depends upon the pressure of the water in the upright pipe and the rate at which the wheel revolves. In giving the theory of the reaction wheel we shall first suppose that there is no work lost in the passage of the water through the pipes or channels from friction, &c. It will be necessary that we should first consider the work accumulated in water at its discharge from a revolving horizontal pipe.

Work accumulated by a Centrifugal Force.

78. Let w=the weight in lbs. of a portion of fluid at the distance x from the axis of motion, v=the angular velocity of the arm in feet per second; r=the radius of the circle described by the orifice at which the fluid is discharged, and V=the velocity of this point; then from eq. (2), Art. **289**, we have

$$\text{Centrifugal force acting on } w \text{ at } x, \text{ or } \text{F}=\frac{v^2wx}{g}.$$

This expression shows that the pressure upon the fluid, resulting from the centrifugal force, varies with the distance from the centre of motion or the vertical axis of the arm. Now let AC represent the arm of the wheel; A the axis; AC=r; AB=x; BD=F the pressure produced at B by the centrifugal force. Through AB draw AE; then this line will be the locus of the pressures. Draw CE parallel to BD; then the area ACE will represent the units of work, u, done by the centripetal force in moving w from A to C (see Art. **33.** p. 24.); that is

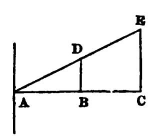


Fig. 35.

$$u=\text{area ACE}$$
$$=\tfrac{1}{2}\text{AC}\times\text{CE},$$

but AC=r, CE=centrifugal pressure at E$=\dfrac{v^2wr}{g}$,

$$\therefore\ u=\tfrac{1}{2}r\times\frac{v^2wr}{g}=\frac{(vr)^2w}{2g},$$

but $vr=\text{V}$,

$$\therefore\ u=\frac{\text{V}^2w}{2g}\ \ldots\ (1):$$

this expression shows, that the velocity of discharge is equal to the velocity of the orifice, and that the pressure of discharge is due to the height $\dfrac{\text{V}^2}{2g}$.

BARKER'S MILL, OR THE REACTION WHEEL.

79. Let v=the velocity of the effluent stream due to the pressure of the water in the vertical pipe as well as that which is due to the centrifugal force; v_1=the absolute velocity of the water after discharge; w=the weight of water discharged per second; U_1=the useful work done per second; h=the height of the water in the vertical pipe above the orifice of discharge; then

Work at efflux=work due to the pressure of the vertical column + work due to the centrifugal force;

$$\therefore\ \frac{v^2 w}{2g}=wh+\frac{\text{V}^2 w}{2g},$$

$$\therefore\ v=\sqrt{\text{V}^2+2gh}.$$

Now the effluent water is projected in a direction contrary to that in which the orifice of the arm is moving,

$$\begin{aligned}\therefore\ v_1&=v-\text{V}\\ &=\sqrt{\text{V}^2+2gh}-\text{V}.\end{aligned}$$

We now have,

Useful work = Work due to the pressure of the vertical column − Work remaining in the water after efflux;

$$\begin{aligned}\therefore\ \text{U}_1&=wh-\frac{v_1^2 w}{2g}\\ &=\frac{w}{g}(\sqrt{\text{V}^2+2gh}-\text{V})\text{V}\ \ldots\ (1).\end{aligned}$$

Here the useful work increases with V, the velocity of the arm; for by the binomial theorem, we have

$$\sqrt{\text{V}^2+2gh}=\text{V}\left(1+\frac{2gh}{\text{V}^2}\right)^{\frac{1}{2}}=\text{V}\left(1+\frac{gh}{\text{V}^2}-\frac{g^2h^2}{2\text{V}^4}+\text{\&c.}\right);$$

$$\begin{aligned}\therefore\ \text{U}_1&=\frac{w}{g}\left(\frac{gh}{\text{V}}-\frac{g^2h^2}{2\text{V}^3}+\text{\&c.}\right)\text{V}\\ &=wh \text{ when } \text{V}=\infty;\end{aligned}$$

that is to say, the useful work is equal to the whole work applied when the velocity is infinite.

But as the velocity of the wheel increases, the prejudicial resistances, arising as well from the friction of the water as the friction upon the axis, also increase; and consequently the maximum effect will take place for some definite velocity, which in general will be found to be considerable. This condition of maximum

efficiency is rather unfavourable to the working of the machine; however for certain mean velocities of rotation, the modulus of the machine does not differ much from that maximum effect, as the following illustrations will show:

Suppose the machine to be loaded so that v^2 shall be equal to $2gh$, or that the velocity of rotation shall be equal to the velocity due to the pressure of the column of water; then eq. (1) becomes

$$U_1 = \frac{w}{g}(\sqrt{4gh} - \sqrt{2gh})\sqrt{2gh}$$

$$= 2(\sqrt{2} - 1)wh = \cdot 828\, wh \quad \ldots (2)$$

where, in this case, ·828 is the modulus.

Supposing $v^2 = 8gh$, we in like manner find the modulus to be ·944.

Hence it appears, that ***these wheels have a considerable modulus, when the velocity of rotation exceeds the velocity due to the height of the fall.***

In the foregoing calculation no allowance has been made for the coefficient of efflux or for the loss of work occasioned by the passage of the water through the pipes or channels. According to the experiments of Morin the modulus of the best reaction wheels does not exceed ·75.

80. In order to diminish the friction upon the axis, the water is sometimes transmitted by a pipe which descends beneath the wheel and then turns vertically upwards. The vertical axle, in this case, is hollow and fits on to the extremity of the supply pipe with a stuffing box. In this construction the upward pressure of the water must be equal to the weight of the wheel, so that the pressure upon the axis may be nothing. To calculate the proportion of the parts of this machine. Let W = the weight of the wheel; r = the radius of the supply pipe; h = the head of water; w = the weight of a cubic foot of water; then we have

$$\text{Upward pressure water} = \pi r^2 h w;$$

but this must be equal to the weight of the wheel,

$$\therefore \pi r^2 h w = W;$$

$$\therefore r = \sqrt{\frac{W}{\pi h w}} \quad \ldots (3).$$

The Centrifugal Pump.

81. In this machine water is raised by means of the centrifugal force given to the water in a curved vane or arm, proceeding from the vertical axis. The dynamical principles of this machine are precisely the same as those of the reaction wheel. But they differ in their objects: in the latter machine a fall of water gives a rotatory motion to a vertical axis; in the former a rotatory motion is given to a vertical axis with the view of elevating a column of water; in both machines the centrifugal action constitutes the characteristic feature. The formula (1), Art. **79.**, given for the reaction wheel, with slight modifications, will apply to the centrifugal pump.

Let U = the work applied per second in giving motion to the vertical axis; h = the height to which the water is raised or the distance between the level of the water in the well and the orifice of discharge; and so on, as in the rotation for the reaction wheel.

Now, in this case, we have

Work at efflux = work due to the centrifugal force − work in raising the water;

hence we find, as in the case of the reaction wheel,

$$v=\sqrt{\text{V}^2-2gh};$$

$$\therefore\ v_1=\sqrt{\text{V}^2-2gh}-\text{V}.$$

We also have

Work applied = work in raising the water + work remaining in the water after efflux;

$$\therefore\ \text{U}=wh+\frac{v_1{}^2 w}{2g}$$

$$=\frac{w}{g}(\text{V}-\sqrt{\text{V}^2-2gh})\text{V}\ \ldots\ (1).$$

In this case the useful work is expressed by $\text{U}_1=wh$. Let M be put for the modulus of the machine; then

$$\text{M}=\frac{\text{U}_1}{\text{U}}$$

$$=\frac{gh}{(\text{V}-\sqrt{\text{V}^2-2gh})\text{V}}\ \ldots\ (2).$$

It may be shown, as in the case of the reaction wheel, that M is a maximum, or equals unity, when V is infinite.

As in the case of the reaction wheel, let $V^2=2gh$; then from eq. (2), we get

$$M=\frac{gh}{2gh}=\cdot 5.$$

Let $V^2=4gh$; then

$$M=\frac{gh}{(\sqrt{4gh}-\sqrt{2gh})\sqrt{4gh}}=\frac{1}{2(2-\sqrt{2})}=\cdot 85.$$

Let $V^2=6gh$; then $M=\cdot 9$.

Hence it appears that *the centrifugal pump has a considerable modulus when the velocity of rotation exceeds the velocity due to twice the height of the column of water raised.* To obtain the same theoretical modulus in this machine as in the reaction wheel, the velocity of the former must be about double the velocity of the latter.

82. The experiments, conducted at the Great Exhibition, on Appold's Centrifugal Pump, with curved arms, gave the maximum modulus ·68. But when the arms were straight and radial, the modulus was as low as ·24; showing the great advantage of having the curved form of the arms, which causes the water to be projected in a tangential direction.

On this subject Professor Moseley observes in his report: —

"If the vanes be straight, it is evident that whatever may be the velocity of the water in the direction of a radius, when it leaves the wheel, its velocity in the direction of a tangent will be that of the circumference of the wheel, so that the greater the velocity of the wheel, the greater will be the amount of vis viva remaining in the water when discharged. and the greater the amount of power uselessly expended to create that vis viva.

"If, however, the vanes be curved backwards, as regards the motion of the wheel, so as to have nearly the direction of a tangent to the circumference of the wheel at the points where they intersect it, then the velocity due to the centrifugal force of the water carrying it over the surface of the vein in the opposite direction to that in which the wheel is moving, and nearly in the direction of a tangent to the circumference, will—if this velocity of the water over the vane in the one direction be equal to that in which the vane is itself moving in the other—produce a state of absolute rest in the water, and entire exhaustion of vis viva. And in whatever degree the equality of these two motions—of the water in one direction over the vane, and of the vane itself in the opposite direction—is attained, in that same degree will the water be

delivered in a state approaching to one of rest. The expedient of curved vanes is adopted in Mr. Appold's pump.

"With regard to the admission of water to the wheel, it is obvious that it should pass directly from the suction-pipe into the wheel without the intervention of any reservoir in which the vis viva of the influent stream—communicated in the act of rising through the pipe—may expend itself, and that such space should be allowed at the centre as not to alter the dimensions of the influent stream. It would further seem expedient, by means of properly-constructed channels, to divide the water into separate streams, and to give to these divergent streams such curvatures as would facilitate their entrance upon the channels formed by the vanes; as in the turbine, or in the reaction wheels.

"It is obvious that the tendency of the centrifugal force continually to increase the velocity of the water *over the vanes* as it recedes from the centre, cannot take effect in respect to all the particles of water in the same section, unless the sections of the channels diminish. If they do not, some of the particles of water in each section must be continually retarded, and power be uselessly expended in producing this retardation; whilst the current cannot but suffer from it a disturbance destructive of its vis viva.

"This diminution of the sections of the channels might probably best be effected by giving to the sides of the wheel the forms of conical discs; an expedient which is adopted in Mr. Lloyd's blowing-machines, and in Mr. Bessemer's centrifugal pump.

"The communication of motion to the water of the reservoir in which the wheel revolves, and into which the water is discharged, should by every practicable expedient be avoided; and for this object the water should be kept as much as possible from the sides of the wheel. This is effected in Mr. Appold's pump, by fixing the wheel between two cheeks which project from opposite sides of the reservoir. The velocity with which the wheel must be driven depends upon the height to which the water is to be raised. Beyond a certain height this velocity is practically unattainable. But long before this limit is reached, it becomes inconsistent with an economical application of the power which drives the pump. It is probably therefore only in comparatively small lifts, where a large quantity of water is to be discharged, that the centrifugal pump will be found useful."

PUMPS.

83. The proportion of the parts of a pump, as in every other

engine, should be determined by the principle of the equality of work, that is, in the present case, the work applied to the piston should be equal to the work in raising the water.

In the common pump. Let A = the area of the piston; l = the length of the stroke; h = the vertical height of the bottom of the barrel above the surface of the water to be raised; $h_1 = h + l$ = the vertical height of the pipe by which the water is discharged from the water to be raised; w = the weight of a cubic foot of water; U = the work applied at each double stroke; m = the modulus of the pump; then by Art. **169.**, page 157.,

$$\text{Work in raising the water into the barrel} = Alw(h + \tfrac{1}{2}l) \dots (1),$$

$$\text{,, \quad ,, \quad from the barrel} = Alw \times \tfrac{1}{2}l;$$

$$\therefore \text{ Total work} = Alw(h + \tfrac{1}{2}l) + Alw \times \tfrac{1}{2}l = Alw(h + l),$$

$$\therefore mU = Alw(h + l) = Alwh_1 \dots (2);$$

$$\therefore l = \frac{mU}{Awh_1} \dots (3),$$

which gives the length of the stroke when U, A, &c., are given.

84. In the case of the forcing pump, let h_1 = the vertical height of the nozzle by which the water is discharged from the bottom of the barrel; then we have

$$\text{Work of the downward stroke} = Alw(h_1 - \tfrac{1}{2}l),$$

which added to eq. (1), gives

$$\text{Total work} = Alw(h + \tfrac{1}{2}l) + Alw(h_1 - \tfrac{1}{2}l)$$
$$= Alw(h + h_1),$$

$$\therefore mU = Alw(h + h_1) \dots (4),$$

$$\therefore l = \frac{mU}{Aw(h + h_1)} \dots (5).$$

85. The pump, although simple in its construction, is far from being an economical machine for raising water. According to the experiments of Morin, its modulus rarely exceeds ·45. This loss of work chiefly depends upon the following causes:

1. On the large size of the barrel of the pump as compared with the suction and force pipes. See Art. **49.**

2. On the form of these pipes where they join with the barrel, or at their extremities. See Arts. **50.** and **31.**

3. On the small size of the valves. See Art. **51.**

4. On the want of a due attention to the proper proportion of the parts of the pump (see Art. **84.**); and the rate at which it is

worked, which occasions a loss of work by giving an unnecessary motion to the water in its passage through the barrel and the pipes. If more work be done upon the piston than is requisite for raising the water, that excess of work becomes accumulated in the water, which in the pump, as it is usually constructed, is lost.

The Pump with Suction Air Chamber.

86. A pump of this kind was exhibited by Mr. Self in the Agricultural Department of the Great Exhibition. The peculiarity of its construction consists in having an air chamber added to a common suction pump; this air chamber communicates with the suction pipe immediately below the barrel. In working the common pump the sudden jerk, which it is desirable to give to the piston at the commencement of the stroke, not unfrequently separates the piston from the water in the barrel, and thereby causes a vacuum to be formed, into which the external air is almost sure to find its way. Now the suction air chamber is calculated to remedy this evil, as well as to save the work accumulated in the water. On this subject Professor Moseley observes in his report: —

"It is immaterial in what proportions this work is distributed over the stroke, or under what varying degrees of pressure it is generated, provided that the pressure never exceeds that of the atmosphere on the surface of the piston. If this pressure be exceeded, the piston may separate itself from the water beneath it in the barrel, the pump drawing air; and this is more likely to occur at the commencement than at any other period of the stroke, the motion of the water at that point being necessarily slow.

"To communicate a finite velocity to the water at the commencement of the stroke, or while the space described by the piston is still exceedingly small, requires a much greater pressure than afterwards; and the greater, as the section of the suction pipe is less, as compared with that of the barrel, and as the lift is greater. Thus at the commencement of the stroke a finite velocity of the piston can only be obtained by an extraordinary effort of the motive power associated with the chance of drawing air and of a shock, if the pressure be suddenly applied. A remedy for some of these evils in the working of a pump has been sought in the application to it of a second air vessel, communicating with the suction pipe immediately below the barrel, or with the top of the suction pipe and the bottom of the barrel. The commencement of each stroke is eased by a supply of water from this air chamber to

the space beneath it. The influx of the water into that space is aided by the pressure of the condensed air in the air chamber, and when the stroke is completed, the state of condensation of this air is, by the momentum of the water in the suction pipe, restored, causing it to rush through the passage by which that pipe communicates with the air chamber. Thus, by this contrivance, the surplus work, which remains in the water of the suction pipe at the conclusion of each stroke, is stored up in the compressed air of the air chamber, and helps to begin the next stroke of the piston.

"The nature of this action will be best understood from that of the hydraulic ram. The contrivance constitutes, indeed, in some respects, a union of the action of the ram with that of the pump; and, besides accomplishing the object for which it was applied, appears to have the effect of considerably economising the power employed in working pumps."

CHAP. XXII.

PRESSURE AND DENSITY OF ELASTIC FLUIDS GENERALLY. WORK IN THE EXPANSION OF ELASTIC FLUIDS, ETC.

Pressure of the Atmosphere.

87. THE pressure of the atmosphere is determined at any time by the height of the column of mercury in the tube of the barometer. (See the Author's Hydrostatics, &c. published in Gleig's Series.)

Let h = the height of the column of mercury; K = the no. sq. in. in the section of the tube; 13568 oz. or 848 lbs. = the weight of a c. ft. of mercury at the mean temperature; p = the no. lbs. pressure of the air on sq. inch of surface; then

$$\text{Pressure of the atmosphere on } K \text{ sq. in.} = p \times K,$$

$$\text{Weight of the column of mercury} = \frac{Kh}{1728} \times 848.$$

But the column of mercury balances the pressure of the atmosphere;

$$\therefore \; p \times K = \frac{Kh}{1728} \times 848,$$

$$p=\frac{848}{1728}\times h=\frac{53}{108}h \ldots (1).$$

$$\therefore\ p=\tfrac{1}{2}h, \text{ very nearly;}$$

hence it appears, that *the pressure of the atmosphere in lbs. per square inch, is very nearly equal to one-half the column of mercury in the barometer tube in inches.*

The column of mercury in the barometer varies in this country from 28 to 31 inches; therefore the mean column is 29·5 inches; and by the foregoing result the mean pressure of the atmosphere will be about 14·7 lbs. per sq. in.

The elasticity or pressure of air is inversely as the space which it occupies.

88. This law of elasticity was first proved by Marriotte in the following manner.

Experiment. Take a bent tube, HEAB, closed at B; introduce a little mercury, so as to make it stand at the same level EA in both legs of the tube; let the space AB, occupied by the inclosed air, be divided into equal parts; pour mercury into the tube until the volume of air in AB is reduced to CD; then it will be found, that when CB is one-half AB, the column of mercury DH, producing this compression, is about 30 inches, or a column of mercury which balances the pressure of the atmosphere; that when CB is one-half AB, or when the volume of air is reduced three times, the column of mercury, DH, is twice 30 inches; and so on, thereby proving the law of elasticity just explained.

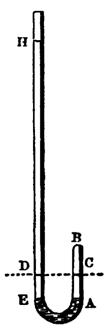


Fig. 36.

89. If P be pressure of a given volume V of air, to find the pressure P_1 of this air when its volume is V_1.

Here by Marriotte's law, we have

Pressure air at V volume $=$ P,

$\therefore$ „ „ 1 „ $=P\times V$,

$\therefore$ „ „ V_1 „ $=\dfrac{P\times V}{V_1}$,

$$\text{that is, } P_1=\frac{P\times V}{V_1} \ldots (1)$$

$$\text{or, } P\times V=P_1\times V_1 \ldots (2).$$

Law of Expansion of Gases by Heat.

90. The following experimental laws were discovered by Gay Lussac and Dalton: 1st. All gases, under the same pressure, expand uniformly for equal increments of temperature; 2nd. The expansion due to the same increase of temperature is the same for all gases; and 3rd. The expansion of a given volume of gas at freezing point, or 32°, is $\frac{1}{490}$th part of this volume for every degree of temperature.

Let v=the volume of gas at 32°, and v=the volume at t degrees; then

$$\text{Increase of temperature above } 32^\circ = t-32.$$

$$\text{Increase of volume for 1 degree} = \frac{v}{490},$$

$$\therefore \quad \text{,,} \quad \text{,,} \quad \text{for } (t-32) \text{ degrees} = \frac{v}{490}(t-32),$$

$$\therefore \ \mathrm{v} = v + \frac{v}{490}(t-32)$$

$$= \frac{v}{490}(458+t) \ldots (1).$$

Let v be the volume of air at t temperature, and v_1 the volume of the same air at t_1 temperature; then

$$\mathrm{v}_1 = \frac{v}{490}(458+t_1);$$

hence we have by division,

$$\frac{\mathrm{v}}{\mathrm{v}_1} = \frac{458+t}{458+t_1} \ldots (2).$$

91. *When the volume of a gas remains the same, its increase of pressure is in proportion to the increase of temperature.*

Let v be the volume of the gas at 32° and p its pressure; v the volume of the same gas at t degrees when the pressure remains the same, and P its pressure when the volume remains the same; then

$$\mathrm{v} = \frac{v}{490}(58+t);$$

but by Marriotte's law, we have

$$v \times \mathrm{P} = \mathrm{v} \times p,$$

and by multiplying these equalities together, and reducing, we get

$$P = \frac{p}{490}(458 + t) \ldots (1).$$

Comparing this with eq. (1), Art. **90.**, it will be seen that the relation of pressures is expressed by the same law as the relation of volumes.

In like manner, we have

$$\frac{P}{P_1} = \frac{458 + t}{458 + t_1} \ldots (2).$$

92. Given the weight w lbs. of a cubic foot of gas at t temperature, to find the weight, w_1, of a cubic foot of the gas when the temperature is t_1.

Let V and V_1 be the same quantity of air at the temperatures t and t_1; then

$$V \times w = V_1 \times w_1, \text{ or } \frac{V}{V_1} = \frac{w_1}{w};$$

substituting in eq. (2), Art. **90.**, and reducing, we get

$$w_1 = \frac{458 + t}{458 + t_1} \times w \ldots (1).$$

Now if we take $1\frac{2}{9}$ oz. as the weight of a cubic foot of air at 60°, the barometer having a mean height, we have by making $t = 60$, and $w = 1\frac{2}{9}$ in eq. (1)

$$w_1 = \frac{518}{458 + t_1} \times 1\frac{2}{9} = \frac{633 \cdot 1'}{458 + t_1} \text{ oz.}$$

$$= \frac{39 \cdot 57}{458 + t_1} \text{ lbs.} \ldots (2),$$

which gives the weight of a cubic foot of air at t_1 temperature.

It has been found from experiment, that the density of steam, in contact with the water from which it is raised, is always $\frac{5}{8}$ of the density of atmospheric air at the same pressure and temperature; hence we have for the weight w_1 of a cubic foot of steam at t_1 temperature

$$w_1 = \frac{5}{8} \times \frac{39\,57}{458 + t_1} \text{ lbs.} \ldots (3)$$

$$= \frac{25}{458 + t_1} \text{ lbs. nearly.}$$

If v_1 be put for the volume of a cubic foot of water in the form of steam at t_1; then

$$v_1 = \frac{\text{wt. c. ft. water}}{\text{wt. c. ft. steam}} = 62 \cdot 5 \div \frac{5}{8} \cdot \frac{39 \cdot 57}{458 + t^1}$$

$$=2{\cdot}527(458+t_1) \dots (4)$$
$$=2\tfrac{1}{2}(458+t_1) \text{ nearly,}$$

which express the number of times that the volume of the steam is greater than the volume of the water from which it is raised. If the water be at the boiling point, or $t_1=212°$, then from eq. (4), we find $v_1=1693$, or 1700 nearly.

93. To find the relative volumes of a gas at different temperatures and pressures.

Let v be the volume of the gas at 32°, and p its pressure; V' the volume of the same gas at t temperature the pressure remaining the same; then

$$V'=\frac{v}{490}(458+t).$$

Now suppose this gas to change its volume from V' to V, and let P be pressure at this new volume, then we have by Marriotte's law

$$P \times V = p \times V',$$

eliminating V' between these equations, we get

$$V=\frac{v}{490}(458+t)\times\frac{p}{P};$$

similarly, we have

$$V_1=\frac{v}{490}(458+t_1)+\frac{p}{P_1},$$

hence we have by division

$$\frac{V}{V_1}=\frac{458+t}{458+t_1}\times\frac{P_1}{P} \dots (1)$$

$$\text{or } \frac{V\times P}{V_1\times P}=\frac{458+t}{458+t_1} \dots (2).$$

In the case of steam, if $t_1=212°$, $P_1=15$, and $V_1=1670$, which is the volume of steam raised from a unit of water at this temperature and pressure; then eq. (3) becomes

$$\frac{V}{1670}=\frac{458+t}{670}\times\frac{15}{P},$$

$$\therefore\ V=37\tfrac{1}{3}\times\frac{458+t}{P}, \text{ nearly} \dots (3),$$

which gives the volume of steam, at P pressure and t temperature, raised from a unit of water.

When the values of P and t are given by experiment, the value of V may be determined from this formula. It is in this way that

tables giving the volume and pressure of steam are usually constructed: For example, let the pressure of steam as determined by experiment, be 100 lbs. when the temperature is 330°, then we have

$$\text{v}=37\tfrac{1}{3}\times\frac{458+330}{100}=294.$$

Work of the Expansion of Gases.

94. The relation of the volume and pressure of gases admits of the following graphic representation.

Suppose the gas to be confined in a cylinder having a piston working air-tight, as in *fig.* 1., p. 6.; let K=the area of the piston in sq. in.; h_1=the height of the piston in the cylinder when the pressure of the gas is P lbs. per sq. in.; x=the height of the piston when the pressure is y lbs.; then

$$\text{the volume at P lbs. pressure}=\text{K}\times h_1,$$

$$\text{,, ,, ,, } y \text{ lbs. ,, } =\text{K}\times x,$$

therefore by Marriotte's law eq. (2), Art. **89.**,

$$y\times\text{K}\times x=\text{P}\times\text{K}\times h_1,$$

$$\therefore\ xy=\text{P}h_1 \ldots (1).$$

Now if x and y be taken as the variable coordinates of a curve, this expression will represent the equation of a *rectangular hyperbola*.

Take $\text{A}x$, $\text{A}y$ as rectangular axes of coordinates, $x=\text{AN}$, $y=\text{NM}$, $h_1=\text{AB}$, $\text{P}=\text{BC}$; then the rectangular hyperbola CMQ will represent the locus of eq. (1), where the axes $\text{A}x$, $\text{A}y$ are the asymptotes to the curve.

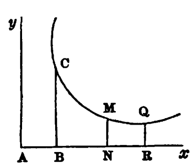


Fig. 37.

95. To find the work of expansion between the pressures P and y.

It has been shown in Art. **33.**, p.24., that the work of expansion between the pressures, $\text{P}=\text{BC}$, and $y=\text{NM}$, is equal to the area of the space BCMN; but by mensuration this area is equal to $h_1\text{P}\log\frac{x}{h_1}$ or $h_1\text{P}\log\frac{\text{P}}{y}$, where log expresses the hyperbolic logarithm;

$\therefore$ Work of expansion upon each sq. in. of the piston $= h_1 P \log \frac{x}{h_1}$ or $h_1 P \log \frac{P}{y} \ldots (1)$.

Let q = the no. c. ft. of gas; P and P_1 the pressures between which the work is done; U_q = the work; then

$$U_q = K h_1 P \log \frac{P}{P_1},$$

$$\text{but } q = \frac{K}{144} \times h_1,$$

$$\therefore\ U_q = 144 q P \log \frac{P}{P_1} \ldots (2),$$

which is the work of q c. ft. of gas, when it expands itself from P to P_1 pressure.

Efflux of Gases.

96. Let the gas be discharged from an orifice with the velocity v ft. per sec.

Put w = the weight of a cubic foot of the gas; q = the volume of gas discharged per second; then

$$\text{Work accumulated in the gas} = \frac{qw \times v^2}{2g};$$

but this work has to be accumulated by the expansion of the gas;

$$\therefore\ \frac{qw \times v^2}{2g} = 144 q P \log \frac{P}{P_1};$$

$$\therefore\ v = 12 \sqrt{\frac{2gP}{w} \log \frac{P}{P_1}} \ldots (1).$$

Or, since $2g = 64\frac{1}{3}$, we have very nearly,

$$v = 96 \sqrt{\frac{P}{w} \log \frac{P}{P_1}} \ldots (2).$$

From the property of logarithms this result may also be expressed by

$$v = 96 \sqrt{\frac{-P}{w} \log \frac{P_1}{P}} \ldots (3).$$

When the pressures differ little from each other, the following approximate formulæ may be employed.

By development, we have

$$\log \frac{\text{P}_1}{\text{P}} = \log \left(1 + \frac{\text{P}_1 - \text{P}}{\text{P}}\right) = \frac{\text{P}_1 - \text{P}}{\text{P}} - \frac{1}{2}\left(\frac{\text{P}_1 - \text{P}}{\text{P}}\right)^2 + \&\text{c}.$$

Neglecting the powers of $\frac{\text{P}_1 - \text{P}}{\text{P}}$ above the first, and substituting in eq. (3), we get

$$v = 96\sqrt{\frac{\text{P} - \text{P}_1}{w}} \ldots (4).$$

Neglecting the powers of $\frac{\text{P}_1 - \text{P}}{\text{P}}$ above the second, we have more accurately

$$v = 96\sqrt{\frac{\text{P} - \text{P}_1}{w}\left(1 + \frac{\text{P} - \text{P}_1}{2\text{P}}\right)} \ldots (5).$$

If h be put for the height of a homogeneous fluid, of the same density as the gas, requisite to produce the pressure $\text{P} - \text{P}_1$ of propulsion; then $144(\text{P} - \text{P}_1) = wh$, and $\frac{\text{P} - \text{P}_1}{w} = \frac{h}{144}$; substituting in eq. (4), we get

$$v = 8\sqrt{h} \ldots (6).$$

It must be observed that this formula is only true when the pressures P and P_1 are nearly equal to each other.

97. *Coefficient of efflux.*—When air issues from an orifice the section of the current undergoes a contraction similar to that observed in the efflux of water.

Let e = the coefficient of efflux, and a = the area of the orifice in sq. ft.; then

$$q = e \times a \times v$$

$$= 12ea\sqrt{\frac{2g\text{P}}{w} \log \frac{\text{P}}{\text{P}_1}} \ldots (7).$$

According to the experiments of Koch, $e = \cdot 58$ when the air issues from an orifice made in a thin plate; $e = \cdot 74$ when the air issues from a pipe about six times as long as it is wide; and $e = \cdot 85$, when the air issues from the conical nozzle of a bellows about five times as long as it is wide and having a lateral convergence of 6°.

To find the velocity of efflux, &c., when the pressure of the air is given in the tube through which it flows.

98. Let P_1 = the pressure of the air in the tube (as indicated by

the air gauge G), and v_1=its velocity; P_2=the pressure of the air at efflux, and v_2=its velocity; P=the pressure of the quiescent air in the receiver; a_1=the section of the tube A; a_2=the section of the orifice O; then

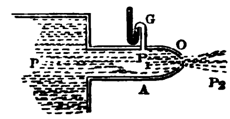


Fig. 38.

Work gained by the air passing from the tube to the orifice of discharge $=\frac{qw}{2g}(v_2^2-v_1^2)$.

Now this work is due to the expansion of the air from P_1 to P_2 pressure: hence we have from eq. (2), Art. **95.**,

$$\frac{qw}{2g}(v_2^2-v_1^2)=144qP_1\log\frac{P_1}{P_2}\ldots(1);$$

but the volume of air passing through the tube per sec. is a_1v_1, and that which passes through the nozzle is a_2v_2: hence we have, by Marriotte's law,

$$a_1v_1\times P_1=a_2v_2\times P_2;\ \therefore\ v_1=\frac{a_2P_2}{a_1P_1}\cdot v_2;$$

substituting this in eq. (1), and reducing, we get

$$v_2=12\left\{\frac{2gP_1\log\frac{P_1}{P_2}}{w\left\{1-\left(\frac{a_2P_2}{a_1P_1}\right)^2\right\}}\right\}^{\frac{1}{2}}\ldots(2),$$

which expresses the velocity of efflux.

The relation of P and P_1, is given in eq. (1), Art. **96.**, by changing v into v_2 and P_1 into P_2.

CHAP. XXIII.

ON THE LAW OF THE EXPANSION OF STEAM. WORK OF STEAM, ETC.

99. WHEN steam is generated in a close vessel, as in the boiler of a steam engine, the density of the steam increases with the temperature, but so long as the temperature continues the same only a certain quantity of steam can be raised from the water, and if the

temperature fall, a portion of the steam resumes the liquid form, and then the density of the steam is diminished. Under these circumstances, the space in the boiler is said to be *saturated* with steam, or that the steam has a *maximum density.*

It has been already stated, that the density of saturated steam is always about five-eighths that of the atmosphere when they are both under the same pressure and at the same temperature.

The relation of the temperature and the pressure of saturated steam has also been determined experimentally, and from that the relation of the volume and pressure has been deduced. (See Table p. 43.) When saturated steam changes its volume, as in the cylinder of a steam engine, it preserves its condition of maximum density, without any deposition of water, provided there is no absolute heat abstracted from it; but if the cylinder should have a less temperature than the steam, a certain quantity of caloric is taken away from the steam, and then a portion of that steam resumes the form of water.

Various empirical formulæ, based upon experimental data, have been given for expressing the relation of the volume and pressure of steam at a maximum density. The most recent of these have been given by Chevalier Pambour and Mr. Pole.

Mr. Pole's formula, although a decided improvement on the Chevalier Pambour's, is not sufficiently accurate for pressures above 70 lbs. or below 16 lbs. Owing to the improvements which have taken place in expansive steam-engines, it becomes necessary to have a formula embracing a much more extensive range of pressures than this. The plan of adopting two formulæ, is of little utility; for in calculating the work of an engine, one formula only can be used, and all properly constructed expansive engines must work with steam having a high pressure at the commencement, and a very low pressure, at the end of the stroke. For this reason, the expressions given for the work of expansive engines, in Pambour's Treatise on the Steam-engine, bring out results which are certainly not sufficiently exact.

In this chapter it is proposed to demonstrate and apply a formula, which the Author some time ago discovered, expressing the law of the expansion of steam; and at the same time to establish certain general equations relative to the work of steam, applicable to all formulæ professing to give the law of volume and pressure.*

* The leading portions of this chapter were first published in the Transactions of the Institution of Civil Engineers, for the year 1848.

The investigation of the formula is instructive, not only as affording an illustration of a general method, which may be used with success for deriving formulæ of a similar kind, but also as giving the scientific man the means of judging how far the formula may be regarded as a mathematical interpretation of the law of expansion. As Mr. Pole had given a table, showing the superiority of his formula over that of the Chevalier Pambour, the Author has only presented a comparative view of his formula and that of Mr. Pole's.

The discussion of the abstract relation of the volume and pressure of steam $V=f(P)$, besides leading to several important general deductions, gives a more scientific character to the mathematical theory of the steam-engine. The principle, which the Author calls the conservation of the work of steam, is a deduction from this general theory. This principle admits of various and interesting illustrations, but on this, as well as other matters connected with the general theory of work of steam, it is not considered expedient to enlarge in this treatise.

Law of the Expansion of Steam.

100. Let V=the volume of a cubic foot of water in the form of steam at P lbs. pressure per square inch; then we have the following relation between V and P:—

$$V=a+bP^{\alpha},$$

where $a=12{\cdot}5$, $b=20570$, and $\alpha=-{\cdot}9301$.

The volume of steam must obviously be some function of its corresponding pressure, that is, $V=f(P)$; and in order to determine this function, assume,

$$V=a+bP^{\alpha}+cP^{\beta}+\&c.,$$

where the coefficients a, b, c, &c., as well as the exponents, α, β, &c. are constants which have to be determined.

In order to find the values of these constants, it is only necessary to substitute a series of corresponding values of V and P, given in experimental tables, and thus obtain as many equations as there are constants in the expression. However, it appears that the determination of three constants gives an expression for V, which assimilates so closely to the relations determined by experiment, as to lead to the inference that it may almost be regarded as an analytical exponent of these experimental data.

Assuming, therefore,

$$V = a + bP^{\alpha} \ldots (1)$$

the next process is to determine the constants a, b, and α.

Let V_1 be the volume of the steam corresponding to P_1 pressure, V_2 the volume corresponding to $4P_1$ pressure, and V_3 the volume corresponding to $16P_1$ pressure, then the assumed relation becomes,

$$V_1 = a + bP_1^{\alpha} \ldots (2)$$
$$V_2 = a + b(4P_1)^{\alpha}$$
$$V_3 = a + b(16P_1)^{\alpha}$$

By transposition these equations become,

$$V_1 - a = bP_1^{\alpha}$$
$$V_2 - a = b(4P_1)^{\alpha}$$
$$V_3 - a = b(16P_1)^{\alpha}$$

Dividing the second by the first, and the third by the second, we get

$$\frac{V_2 - a}{V_1 - a} = 4^{\alpha} \ldots (3); \text{ and } \frac{V_3 - a}{V_2 - a} = 4^{\alpha};$$

$$\therefore \frac{V_2 - a}{V_1 - a} = \frac{V_3 - a}{V_2 - a}.$$

Hence by solving this equation, we get

$$a = \frac{V_1 V_3 - V_2^2}{V_1 + V_3 - 2V_2};$$

which gives the value of the constant a, in terms of the known tabular volumes V_1, V_2, and V_3.

From equation (3),

$$\text{Log. } 4 \times \alpha = \log \frac{V_2 - a}{V_1 - a}$$

$$\therefore \alpha = \frac{\log (V_2 - a) - \log (V_1 - a)}{\log 4}.$$

This expression gives the value of α, the exponent of P, in terms of V_1, V_2, and a which has just been determined.

Then from equation (2),

$$b = \frac{V_1 - a}{P_1^{\alpha}}; \therefore \log b = \log (V_1 - a) - \alpha \log P_1.$$

Now, since a and α are known, the value of b may be readily found from this expression.

In making the calculations, P_1 is taken $=5$ lbs., and therefore, $V_1=4617$; then $4P_1=20$ lbs., giving 1281 for V_2; and $16P_1=80$ lbs., giving 362 for V_3. Substituting these values in the equations for a, α, and b, we find $a=12{\cdot}5$, $b=20570$, and $\alpha=-{\cdot}9301$.

The following Table will show how very nearly the volumes, calculated by this formula, coincide with those derived from experiment.

P in pounds per square inch.	V derived from Experiment.	V calculated from the proposed Formula.	V calculated from Pole's Formula.	Errors of the proposed Formula.	Errors of Pole's Formula.
5	4617	4617	4915	0	+298
15	1669	1668	1681	−1	+ 12
16	1573	1573	1580	0	+ 7
60	470	469	470	−1	0
70	408	408	411	0	+ 3
150	205	206	226	+1	+ 21

101. As an application of this formula, let it be required to find the volume of a cubic foot of water in the form of steam at 16 lbs. pressure per square inch.

Here $P=16$, $\therefore V=12{\cdot}5+20570\times 16^{-{\cdot}9301}$.

In order to calculate the latter part of this expression, take $\log(20570\times 16^{-{\cdot}9301})=\log 20570-{\cdot}9301\log 16=\log 1560{\cdot}5$.

$$\therefore V=12{\cdot}5+1560{\cdot}5=1573 \text{ c. ft.}$$

When V is given to find P,

$$P=\left(\frac{V-a}{b}\right)^{\frac{1}{\alpha}}\ldots(4)$$

$$\therefore \text{Log. } P=\frac{1}{\alpha}\{\log(V-a)-\log b\}.$$

Let $V=325$, then,

$\log P=\dfrac{1}{-{\cdot}9301}\{\log(325-12{\cdot}5)-\log 20570\}=\log 90{\cdot}1$, that is, $P=90{\cdot}1$ lbs.

WORK OF STEAM CONSIDERED IN RELATION TO THE ABSTRACT FORMULA $V=f(P)$.

102. Let U_s be put for the work performed by s cubic feet of

water, in the form of steam, between the pressures P and P_1; and in order to discuss the subject in its most general form, let $ABmn$ be the section of the space in which the steam expands itself; $ABsp$ the original volume of the steam at P pressure; $ABec$ the space V_1 occupied by the steam at P_1 pressure; A ft. the area of the section at ce; dv the distance between the sections ce and nm, taken indefinitely near to each other; then the element of work, or $dU_1 = 144 A \times P_1 \times dv$;

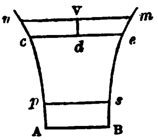


Fig. 39.

$$\text{but } A \times dv = dV_1;$$

$$\therefore\ dU_1 = 144 P_1 dV_1 = 144 P_1 df(P_1).$$

Integrating between the limits P and P_1, we have,

$$U_1 = 144 \int_{P_1}^{P} \{P_1 df(P_1)\}$$

Multiplying each side of the equality by s,

$$U_s = 144 s \int_{P_1}^{P} \{P_1 df(P_1)\} \ldots (1).$$

When the function $f(P_1)$ is known, this expression may be integrated, and then it is obvious that the resulting expression will only contain the elements s, P, and P_1, with certain numerical constants depending upon the form of the function $f(P)$.

Principle of the Conservation of the Work of Steam.

103. Since the preceding expression for U_s shows that the work is entirely independent of the equation of the curve Apn, it follows, that the work of steam between any given pressures P and P_1 is always the same, whatever may be the nature of the space through which the steam expands itself, or whatever may be the form of the law expressing the relation between the volume and pressure of steam.

The Work increases with the Pressure.

104. It also follows from the preceding expression, that, other things being the same, the work will increase with the pressure P, at which the steam is generated; and since the quantity of fuel

necessary to evaporate a given weight of water, does not at all depend upon the pressure under which the steam is raised, it follows that it is most economical to employ steam of as high a temperature as possible.

Efflux of Steam.

105. Let A ft.=the section of an orifice from which steam discharges itself with the velocity of v ft. per second; w=the weight of water evaporated per second; and P_1=the pressure of the steam at the point of efflux; then, the work accumulated in the steam per second,

$$\text{or } U_s = \frac{w \times v^2}{2g}.$$

$$\text{Now } s = \frac{w}{62{\cdot}5},\ V = f(P),\ \therefore sV_1 = sf(P_1) = \frac{w}{62{\cdot}5} f(P_1);$$

$$\text{but } sV_1 = Av;\ \therefore Av = \frac{w}{62{\cdot}5} f(P_1);$$

$$\therefore v = \frac{w}{62{\cdot}5A} f(P_1) \ldots (1).$$

Substituting this value of v in the expression for the accumulated work,

$$U_s = \frac{w^3}{7812{\cdot}5A^2 g}\{f(P_1)\}^2 \ldots (2).$$

Here as $f(P_1)$ is independent of w, it follows that the *theoretical work of steam discharging itself from a nozzle varies as the cube of the water evaporated.*

106. The work accumulated in the steam at efflux is due to the work of expansion between the pressures P and P_1, and as it is no matter (Art. **102**.,) through what peculiar form of vessel this expansion takes place, we have by eq. (1), Art. **102**., the identity,

$$\frac{w \times v^2}{2g} = 144s\int_{P_1}^{P}\{P_1\, df(P_1)\},$$

and by substituting $\frac{w}{62{\cdot}5}$ for s, this equation becomes

$$\frac{w \times v^2}{g} = \frac{288w}{62{\cdot}5}\int_{P_1}^{P}\{P_1\, df(P_1)\} \ldots (3),$$

which is an expression for the *vis viva* of the steam.

107. From eq. (3), we get

$$v=12e\left[\frac{2g}{62\cdot5}\int_{P_1}^{P}\{P_1 df(P_1)\}\right]^{\frac{1}{2}}\ldots(4),$$

which gives the velocity of efflux, e being put for the coefficient of efflux. (See Art. **97.**)

To find the velocity of efflux, &c. when the pressure of the steam is given in the tube through which it flows. See Art. **98.**

108. Here, putting w=the weight of s cubic feet of water evaporated per second, we have from Art. **98.**,

$$U_s=\frac{w}{2g}(v_2^2-v_1^2)=\frac{62\cdot5s}{2g}(v_2^2-v_1^2)\ldots(1).$$

Now let V_1, V_2, be put for the volumes of a cubic foot of water at P_1 and P_2 pressures respectively; then

$$sV_1=a_1v_1,\ sV_2=a_2v_2,$$

$$\therefore\ v_1=\frac{sV_1}{a_1},\ \text{and}\ v_2=\frac{sV_2}{a_2},$$

substituting in eq. (1), and reducing, we get,

$$U_s=\frac{62\cdot5s^3}{2g}\left\{\left(\frac{V_2}{a_2}\right)^2-\left(\frac{V_1}{a_1}\right)^2\right\}\ldots(2),$$

where V_1 and V_2 may be eliminated by the formula $V=f(P)$.

Here again it will be observed, that *the work varies as the cube of the water evaporated.*

Solving this equation for the value of $\frac{sV_2}{a_2}$ or v_2, we get,

$$v_2^2=\frac{2gU_s}{62\cdot5s}+\left(\frac{V_1}{a_1}\right)^2s^2\ldots(3),$$

which gives the velocity of efflux. It will be observed that U_s is given in eq. (1), Art. **102.**, and that V_1 may be eliminated by the formula $V_1=f(P_1)$.

Work of Steam in the Expansive Engine.

109. In order to investigate a general formula expressing the work of steam in an expansive engine, let K=the area of the piston in square inches; h_1=the length of the stroke, including the clearance c; h=the point of the cylinder at which the steam is cut off; s=the number of cubic feet of water evaporated per minute; P=the pressure of the steam at the commencement of the stroke; P_1=the pressure of the steam at the end of the stroke; and N=the number of strokes performed by the piston per minute;

then, assuming as before, $V = f(P)$, the volume of steam discharged per minute at P pressure, is $sV = sf(P)$;

but $sv = \frac{NKh}{144}$;

$$\therefore\ sf(P) = \frac{NKh}{144} \ldots (1),$$

and similarly,

$$sf(P_1) = \frac{NKh_1}{144};$$

$$\therefore \frac{f(P_1)}{f(P)} = \frac{h_1}{h}, \text{ and } f(P_1) = \frac{h_1}{h} f(P) \ldots (2).$$

The work performed upon the piston before the steam is cut off $= NKP(h-c)$; and, as the work performed by the expansion of the steam is given in eq. (1), Art. **102.**, the total work performed by the steam per minute is

$$U_s = 144s \int_{P_1}^{P} \{P_1 df(P_1)\} + NKP(h-c);$$

but by eq. (1), $NK = \frac{144sf(P)}{h}$,

$$\therefore\ U_s = 144s\left[\int_{P_1}^{P} \{P df(P)\} + \frac{h-c}{h} P f(P)\right] \ldots (3),$$

where P, in the final expression may be eliminated by eq. (2).

Now let the work of the resistances opposed to the work of the steam be considered, and take L = the useful load in lbs. upon each inch of the piston; F = the friction in lbs. per inch of the piston, arising from the motion of the unloaded piston; f_1 = the coefficient of friction arising from the useful load; and p = the pressure of the steam in the condenser; then the total resistance upon the piston $= K\{F + L(1+f_1) + p\}$, and therefore the work expended per minute in overcoming this resistance $= (h_1 - c)NK \{F + L(1+f_1) + p\} \ldots (4)$.

Now it is obvious, that when the motion of the piston of the engine attains a certain mean uniformity, the work of the resistances will be equal to the work of the steam; therefore by equating eq. (3) and (4), and substituting $\frac{h}{f(P)}$ for $\frac{144s}{NK}$, we have

$$(h_1 - c)\{F + L(1+f_1) + p\} = \frac{h}{f(P)} \int_{P_1}^{P} \{P df(P)\} + (h-c)P \ldots (5)$$

From this equation the value of the useful load L is readily determined, and then,

$$\text{the useful H.P.}=\frac{\text{NKL}(h_1-c)}{33000}=\frac{144 s f(\text{P})\text{L}(h_1-c)}{33000 h} \ldots (6).$$

Equation (5) gives the general relation of the elements of the problem.

Work of Steam considered in relation to the Formula $V=a+bP^{\alpha}$.

110. In the preceding general expressions it is necessary now to substitute the value of $f(\text{P})$, that is, $f(\text{P})=a+b\text{P}^{\alpha}$.

By differentiating, $df(\text{P})=\alpha b\text{P}^{\alpha-1}d\text{P}$.

Equation (1), Art. **102**., then becomes,

$$\text{U}_s=144 s\int_{\text{P}_1}^{\text{P}}\alpha b\text{P}^{\alpha}d\text{P}=\frac{144\alpha b s}{\alpha+1}\{\text{P}_1^{\alpha+1}-\text{P}^{\alpha+1}\}$$

111. Equation (3), Art. **109**., becomes,

$$\text{U}_s=144 s\left[\frac{\alpha b}{\alpha+1}\{\text{P}_1^{\alpha+1}-\text{P}^{\alpha+1}\}+\frac{h-c}{h}\text{P}(a+b\text{P}^{\alpha})\right] \ldots (1).$$

where, by eq. (2), Art. **109**., P_1 may be eliminated. Thus,

$$a+b\text{P}_1^{\alpha}=\frac{h_1}{h}(a+b\text{P}^{\alpha}), \text{ and } \therefore \text{P}_1^{\alpha+1}=\left\{\frac{h_1(a+b\text{P}_{\alpha})-ah}{hb}\right\}^{\frac{\alpha+1}{\alpha}} \ldots (2).$$

In order to facilitate the numerical calculations, it is necessary to bear in mind, that the expression $a+b\text{P}^{\alpha}$ is the volume of steam from a cubic foot of water at P pressure, which may at once be obtained from a table of volumes and pressures.

112. Equation (1), Art. **109**., becomes,

$$s(a+b\text{P}^{\alpha})=\frac{\text{NK}h}{144} \ldots (1).$$

$$\therefore \text{N}=\frac{144 s(a+b\text{P}^{\alpha})}{\text{K}h};$$

which expresses the number of strokes in terms of the water evaporated, &c.

From eq. (1),

$$\text{P}=\left(\frac{\text{NK}h-144 a s}{144 s b}\right)^{\frac{1}{\alpha}} \ldots (2).$$

and similarly, $P_1 = \left(\frac{NKh_1 - 144as}{144sb}\right)^{\frac{1}{a}} \dots (3).$

These two formulæ express the pressure of the steam in the cylinder, at the commencement and at the end of the stroke, in terms of the number of strokes and the water evaporated.

113. Equation (5), Art. **109.**, becomes

$$(h_1-c)\{F+L(1+f_1)+p\} = \frac{abh}{(a+1)(a+bP^a)}\{P_1^{a+1}-P^{a+1}\}+(h-c)$$
$$P\dots(1),$$

where P_1^{a+1} may be eliminated by eq. (2), Art. **111.**

From this general relation of the elements of the problem the value of the useful load is readily determined; then eq. (6), Art. **109.**, becomes,

$$\text{Useful H.P.} = \frac{144s(a+bP^a)L(h_1-c)}{33000h}\dots(2).$$

When N and s are the given elements in the problem, the following general relation is found by substituting in eq. (1), the values of P and P_1 given in the equations (1), (2), and (3), Art. **112.**, or,

$$(h_1-c)\{F+L(1+f_1)+p\} = \frac{144abs}{(a+1)NK}\left\{\left(\frac{NKh_1-144as}{144sb}\right)^{\frac{a+1}{a}} - \right.$$
$$\left.\left(\frac{NKh-144as}{144sb}\right)^{\frac{a+1}{a}}\right\} + (h-c)\left(\frac{NKh-144as}{144sb}\right)^{\frac{1}{a}}\dots(3).$$

Work of Steam considered in relation to Marriotte's Formula.

114. According to this formula $f(P)$ varies as $\frac{1}{P}$, or by eq. (2), Art. **109.**, $P\times h = P_1\times h_1$. In this case we find as in eq. (1), Art. **95.**,

Work done expansively on 1 inch of the piston in 1 stroke $= hP \log \frac{P}{P_1}$, or $hP \log \frac{h_1}{h}$.

Hence we have from eq. (5), Art. **109.**,

$$(h_1-c)\{F+L(1+f_1)+p\} = hP\log\frac{P}{P_1}+(h-c)P\dots(1).$$

Maximum Velocity of the Piston.

115. By Art. **286.**, page 237., this will take place at that point of the stroke where the pressure of the steam is equal to the sum of all the resisting pressures. But, eq. (3), Art. **40.**, the sum of all the resistances on the piston is equal to the mean pressure of the steam (see eq. (4), Art. **45.**); therefore, *the pressure of the steam at the point of maximum velocity is equal to the mean pressure of the steam.* In order to determine this point; let h'=its height from the bottom of the cylinder; P'=the mean pressure of the steam; U=the work of the steam on 1 inch of the piston in 1 stroke, as given in eq. (4), p. 54., and in the right-hand member of eq. (1), Art. **113.**, &c.; then,

$$\text{P}'=\frac{\text{U}}{h_1}\ldots(1);$$

by eq. (2), Art. **109.**,

$$\frac{h'}{h}=\frac{f(\text{P}')}{f(\text{P})}=\frac{\text{tabular vol. at P}' \text{ pressure}}{\text{tabular vol. at P pressure}}\ldots(2).$$

See page 41. Or taking the Author's formula, $\frac{h'}{h}=\frac{a+b\text{P}'}{a+b\text{P}^a}$

Or taking Marriotte's formula, $\frac{h'}{h}=\frac{\text{P}}{\text{P}'}$.

Whence h' is determined, the value of P' being given in eq. (1).

In order to find the maximum velocity v of the piston; let U'= the work done by the steam up to the point of maximum velocity as given in Art. **109.**, &c.; w=the weight of the whole mass in motion referred to the piston as determined in Art. **252.**, page 216.; then,

Work accumulated in w=work steam − work resistances,

$$\therefore\ \frac{wv^2}{2g}=\text{U}'-\text{P}'\text{K}(h'-c)\ldots(3),$$

whence v may be found. It will be observed, that P' is given in eq. (1), and h' in eq. (2).

Maximum Work of Steam in the Condensing Engine.

116. The steam will perform the greatest amount of work, when the work of the useful load L is a maximum; and this maximum

condition will obviously depend upon the extent to which the expansion of the steam is carried.

From eq. (5), Art. **109.**, we get,

$$(h_1-c)\mathrm{L}(1+f_1)+(h_1-c)(\mathrm{F}+p)=\frac{h}{f(\mathrm{P})}\int_{\mathrm{P}_1}^{\mathrm{P}}\{\mathrm{P}_1 df(\mathrm{P}_1)\}+(h-c)\mathrm{P}.$$

Taking P, h, c, &c. as constant, and P_1, h_1, as the variables; and differentiating with respect to P_1, we get,

$$\frac{d(h_1-c)\mathrm{L}(1+f_1)}{d\mathrm{P}_1}+\frac{d(h_1-c)(\mathrm{F}+p)}{d\mathrm{P}_1}=\frac{h}{f(\mathrm{P})}\cdot\frac{\mathrm{P}_1 df(\mathrm{P}_1)}{d\mathrm{P}_1}.$$

Now, taking $\frac{d(h_1-c)\mathrm{L}}{d\mathrm{P}_1}=0$, so as to render $(h_1-c)\mathrm{L}$ a maximum, we get,

$$\frac{dh_1}{d\mathrm{P}_1}(\mathrm{F}+p)=\frac{h}{f(\mathrm{P})}\cdot\frac{\mathrm{P}_1 df(\mathrm{P}_1)}{d\mathrm{P}_1};$$

but by eq. (2), Art. **109.**, we have,

$$\frac{f(\mathrm{P}_1)}{f(\mathrm{P})}=\frac{h_1}{h}, \quad \therefore \frac{h}{f(\mathrm{P})}\cdot\frac{\mathrm{P}_1 df(\mathrm{P}_1)}{d\mathrm{P}_1}=\mathrm{P}_1\cdot\frac{dh_1}{d\mathrm{P}_1};$$

hence we have by equality,

$$\frac{dh_1}{d\mathrm{P}_1}(\mathrm{F}+p)=\mathrm{P}_1\cdot\frac{dh_1}{d\mathrm{P}_1},$$

$$\therefore \mathrm{P}_1=\mathrm{F}+p \ldots (1);$$

that is to say, THE PRESSURE OF THE STEAM AT THE END OF THE STROKE IS EQUAL TO THE SUM OF THE RESISTANCES OF THE UNLOADED ENGINE, WHATEVER MAY BE THE LAW EXPRESSING THE RELATION OF THE VOLUME AND PRESSURE OF STEAM.

This simple and general theorem, the Author believes, is here given for the first time.

Substituting this value of P_1 in eq. (2), Art. **109.**, we get,

$$\frac{h_1}{h}=\frac{f(\mathrm{F}+p)}{f(\mathrm{P})}=\frac{\text{tabular vol. at } \mathrm{F}+p \text{ pressure}}{\text{tabular vol. at } \mathrm{P} \text{ pressure}} \ldots (2);$$

whence the value of h_1, the length of the stroke, is determined, so as to secure the maximum work.

These values of P_1 and h_1 substituted in eqs. (3) and (5), Art. **109.**, and in eqs. (1), (2), and (3), Art. **113.**, give the various relations for maximum efficiency.

THE END.

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